ALMOST PERIODIC FUNCTIONS

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INTRODUCTION

1. In planning the lectures which form the content of this book I had to make a choice which at first confronted me with a certain difficulty. Since the amount of time in which I had to give the lectures was rather limited, I had to decide in advance whether I wanted to give a more survey-like report on the whole theory or whether I should present only one part - but an essential one - of the theory, this, however, in fullest detail and with complete proofs. I decided for the latter alternative, holding the opinion that the lectures will be more permanently profitable for those attending if they are given the opportunity to assimilate the subject matter without haste - even if this means that the amount of material covered has to be somewhat reduced. The reduction of material was done mainly by dealing only with functions of a <u>real</u> variable, and besides only with <u>continuous</u> functions throughout.

2. In order to make up to some extent for the aforementioned reduction of material I have added two appendices to the lectures proper, as has already been said in the preface. The first of these deals with the <u>generalization</u> of the theory of almost periodic functions (along the lines of Lebesgue's theory of the integral), while the second gives an account of the theory of almost periodic functions of a <u>complex</u> variable. This latter theory by the way, which has grown out of the theory of Dirichlet's series, formed the author's original starting point for the entire investigation.

For the reader who wishes to go deeper into the theory there is a short bibliography at the end, listing some of the pertinent works on the subject.

3. Before starting with the presentation of the theory proper we make some preliminary remarks concerning the problems taken up in the theory of almost periodic functions. Later on, of course, there will be given a detailed and precise formulation of these problems. Generally speaking we can say that the main problem of the theory consists in finding those functions $f(x)$ of the real variable x which are defined for $-\infty < x < \infty$ and which can be resolved into <u>pure</u>

vibrations. This statement contains several words whose meanings must first of all be defined. What are "pure vibrations," and what is to be understood by the word "resolved?" At this place I shall not go any deeper into the concept of resolution, which is most closely connected with the whole structure of the theory. However, I shall now explain precisely what is meant by a pure vibration.

4. As long as only real functions are considered, the term "pure vibration" applies to such functions as $\cos x$, $\sin x$ or more generally $\alpha\cos x+\beta\sin x$; these functions are periodic of period 2π; if arbitrary periods are admitted, then "pure vibration" will stand for a function of the form $\alpha\cos\lambda x+\beta\sin\lambda x$. In the following it will be more convenient for formal reasons to consider not real, but complex functions of the real variable x. By a pure vibration is then meant any function of the form $ae^{i\lambda x}=a(\cos\lambda x+i\sin\lambda x)$, where a denotes an arbitrary complex number and λ an arbitrary real number. If a is written in the form $a=|a|e^{iv}$, the function takes on the form $|a|e^{iv}e^{i\lambda x}$. Here $|a|$ gives the amplitude, v the phase and λ the frequency of the vibration. (The period of the vibration is (for $\lambda\neq 0$) $2\pi/|\lambda|$). Of these numbers, $|a|$ and λ are the most important; frequently $|a|^2$ is considered instead of $|a|$ itself.

5. The theory of almost periodic functions now deals with the following problem: Which functions $f(x)$ can be resolved into pure vibrations on $-\infty<x<\infty$ i.e., are "representable" by a trigonometric series of the form $\sum A_n e^{i\Lambda_n x}$? (We are here concerned only with series having at most denumerably many elements, or physically speaking, with functions whose "spectrum" is a pure line spectrum; the theory of the Fourier integral $\int_{-\infty}^{\infty} a(\lambda)e^{i\lambda x}d\lambda$, i.e., of the continuous spectra, is, therefore not, on principle, within the scope of our problem).

In the classical case where only harmonic vibrations are considered, i.e., vibrations of the form $ae^{in\alpha x}$, $n=0$, $\pm1,\pm2,\ldots$ the corresponding question leads to the theory of the ordinary trigonometric series $\sum_{-\infty}^{\infty} a_n e^{in\alpha x}$; any periodic function of period $p=2\pi/|\alpha|$ can, as is well known, be developed into such a series, the Fourier series of the func-

tion. The essential difference between this case and the above case where arbitrary vibrations $ae^{i\lambda x}$, $-\infty < \lambda < \infty$ were considered, lies in the fact that in the latter case the set of the frequencies which occur is non-denumerable, whereas in the first case we are from the very start dealing only with denumerably many frequencies.

The lectures are divided into two parts. In the **first part** we take up the classical case of harmonic vibrations $ae^{in\alpha x}$, choosing $\alpha = 1$ for simplicity, so that we are dealing with the theory of ordinary Fourier series of period 2π This part is essentially of an introductory character and is intended mainly to make for an easier and deeper understanding of the following part; because of this, the exposition differs from the usual one in some respects.

In the second part there follows a treatment of the arbitrary vibrations $ae^{i\lambda x}$, which leads to the general theory of almost periodic functions and of their Fourier series.

6. Bohl's theory of "functions periodic in the more general sense" occupies an intermediate place between the theory of periodic and the theory of almost periodic functions. This theory corresponds to the case where only vibrations of the form $ae^{i(n_1\alpha_1 + n_2\alpha_2 + \cdots + n_m\alpha_m)x}$ are considered, where $\alpha_1, \alpha_2, \ldots \alpha_m$ are given linearly independent constants while $n_1, n_2, \ldots, n_m$ run independently through all integers $0, \pm 1, \pm 2, \cdots$ Every function periodic in the more general sense with the "periods" $p_1 = 2\pi/|\alpha_1|$, $p_2 = 2\pi/|\alpha_2|, \ldots, p_m = 2\pi/|\alpha_m|$ can be developed into a Fourier series of the form

$$\sum a_{n_1, n_2, \ldots, n_m} e^{i(n_1\alpha_1 + n_2\alpha_2 + \cdots + n_m\alpha_m)x}$$

The beautiful investigations by Bohl are closely related to the theory of almost periodic functions at many points; in a certain sense, however, these Bohl functions are nearer to the pure periodic functions than to the almost periodic ones, the reason being essentially that here, again, the basic system of frequencies is denumerable.

7. In the course of building up the theory, there presented itself, besides the Bohl functions just mentioned, another simple and important subclass, viz., the class of limit-periodic functions: These are the functions which can be developed into trigonometric series of the form $\sum a_r e^{irax}$ where r assumes all rational values. For the sake of brev-

ity, however, I could not in the present lectures enter into a discussion of either Bohl's functions or the limit-periodic functions.

Also of necessity entirely omitted from the present publication are investigations such as those by Franklin and Bochner on almost periodic functions of several - even of denumerably many - variables; the investigations on the problem of distribution of values, a problem which has been attacked from different directions by Wintner and Jessen; also the investigations of differential equations with almost periodic coefficients, carried out mainly by Favard (who in this followed Bohl).

8. The theory of almost periodic functions was developed in its main features by the author in three rather long papers in the Acta Mathematica (Volumes 45, 46 and 47) under the common title "Zur Theorie der Fast periodische Funktionen"; the first of these deals with the almost periodic functions of a real variable, while the third takes up the case of a complex variable.

In building the theory there was encountered a particular difficulty of intrinsic character; this is the decision on "completeness" of the system of all functions $e^{i\lambda x}$. The original proof that this system is indeed a complete one (in a sense to be made precise later) was very complicated and involved many conclusions; its guiding idea, however, was a rather simple one; it was the idea, generally speaking, that the set of all purely periodic functions (with arbitrary periods) can be considered in a certain sense as "being everywhere dense" within the general class of the almost periodic functions, so that from the very start it appeared quite possible to derive the validity of the completeness theorem by means of a limiting process from its well known validity for purely periodic functions (of a given period, i.e., for the system of harmonic vibrations $e^{in\alpha x}$).

Wiener succeeded in finding a new proof of this fundamental theorem, much shorter than mine but, on the other hand, using results of the Lebesgue integral theory and the theory of Fourier integrals; whereas the original proof was worked by quite elementary means. This proof by Wiener acquired special interest by forming an important starting point for his beautiful and promising investigations on combined Fourier series and Fourier integrals.

An important and interesting viewpoint which also leads to a new proof of the fundamental theorem was introduced by Weyl in his discovery of the connections between the theory of almost periodic functions and the theory of integral equations, or rather, mean value equations. The use of considerations from group theory is also characteristic of Weyl's method, which was later simplified somewhat by Hammerstein.

The proof which I shall present in these lectures is, however, neither Wiener's nor Weyl's, but a third one due to De La Vallee Poussin which is close to mine in principle - although essential use is also made of ideas due to Weyl - but which is incomparably simpler and can hardly be improved on in its brevity and elegance. This proof by De La Vallée Poussin, too, starts from the validity of the completeness theorem for pure periodic functions, which theorem is easy to prove and is discussed at length in the first part of these lectures.

9. Although the fundamental theorem constitutes the decisive theorem of the whole theory, it should not be considered the "main theorem" proper; rather, the main theorem is the so-called approximation theorem - the analogue of Weierstrass' classical approximation theorem for pure periodic functions, - which gives the very characteristic property of the almost periodic functions - that of resolvability.

The extant proofs of this approximation theorem are all based in principle on the fundamental theorem. Since my original proof (which is based on a relation essential for certain problems between the theories of almost periodic and of limit-periodic functions of infinitely many variables) there have also here been obtained remarkable simplifications and new developments, mainly by Bochner and Weyl.

In the lectures I have followed Bochner's proof which is outstanding as a beautiful and natural generalization of the Fejer summability method for Fourier series of purely periodic functions. (This latter is dealt with in the first part in detail).

10. In closing I add a few remarks on certain details of the present exposition.

The rather deep proof of the theorem on the integration

of an almost periodic function derives from Bohl; in fact it was possible to carry over this proof - as well as the theorem itself - almost literally from the Bohl functions to the general case of almost periodic functions.

The very last theorem of the entire series of lectures which expresses a particular simplicity in the behavior of a Fourier series $\sum A_n e^{i\Lambda_n x}$ with linearly independent exponents Λ_n, was proved originally by using theorems on Diophantine approximations. The extremely simple proof presented here, which is based on ideas of Bochner and Szidon, I owe to a kind communication from Mr. Fekete.

PURELY PERIODIC FUNCTIONS
AND THEIR FOURIER SERIES

GENERAL ORTHOGONAL SYSTEMS

11. Let $\varphi(x)$ and $\psi(x)$ be two continuous non-vanishing functions defined in the interval $a \leqq x \leqq b$.

When the functions $\varphi(x)$ and $\psi(x)$ are both real, they will be said to be <u>orthogonal to each other</u> if

$$\int_a^b \varphi(x)\,\psi(x)\,dx = 0 .$$

Thus, for example, if m and n are positive integers and $m \neq n$, the two functions $\cos mx$ and $\cos nx$ are orthogonal to each other in $0 \leqq x \leqq 2\pi$, since

$$\int_0^{2\pi} \cos mx \cos nx\,dx = \int_0^{2\pi} \tfrac{1}{2}\{\cos(m+n)x + \cos(m-n)x\}\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{2\pi} = 0 .$$

When $\varphi(x)$ and $\psi(x)$ are complex functions, they will be said to be orthogonal to each other in $a \leqq x \leqq b$ if,

$$\int_a^b \varphi(x)\,\overline{\psi(x)}\,dx = 0 \quad \left(\text{or} \quad \int_a^b \overline{\varphi(x)}\,\psi(x)\,dx = 0 \right).$$

Here a horizontal stroke above the quantity changes it to its complex conjugate value. Thus, if m and n denote arbitrary integers and $m \neq n$, the two functions e^{imx} and e^{inx} are orthogonal to each other in $0 \leqq x \leqq 2\pi$, since

$$\int_0^{2\pi} e^{imx} e^{-inx}\,dx = \int_0^{2\pi} e^{i(m-n)x}\,dx = \left[\frac{e^{i(m-n)x}}{i(m-n)}\right]_0^{2\pi} = 0 .$$

In the two examples just cited the functions considered are periodic, with period 2π. From the orthogonality property in the interval $0 \leqq x \leqq 2\pi$ we can deduce orthogonality in an arbitrary interval $a \leqq x \leqq a + 2\pi$ of length 2π.

In what follows, the functions under consideration are <u>complex</u> unless we specifically state the contrary.

12. It is often convenient to consider in place of the integral $\int_a^b \cdots$ the meanvalue $\frac{1}{b-a}\int_a^b \cdots$ Letting $\varphi(x)$ be any continuous function defined $a \leqq x \leqq b$ we make use of the notation

$$\frac{1}{b-a}\int_a^b \varphi(x)\,dx = M\{\varphi(x)\} = M\{\varphi\}.$$

The function $\varphi(x)$ will be said to be normalized (in the interval $a \leqq x \leqq b$) if the mean value is unity, or, if

$$\frac{1}{b-a}\int_a^b \varphi(x)\,\overline{\varphi(x)}\,dx = \frac{1}{b-a}\int_a^b |\varphi(x)|^2\,dx = 1.$$

Thus, for example, if n is an integer, the function e^{inx} is normalized in $0 \leqq x \leqq 2\pi$, and in fact even in any interval $a \leqq x \leqq b$, since

$$\int_a^b e^{inx}\,e^{-inx}\,dx = \int_a^b 1\,dx = b-a.$$

On the other hand, the function $\cos nx$, where n denotes a positive integer, is not normalized in $0 \leqq x \leqq 2\pi$, since

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^2 nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+\cos 2nx}{2}\,dx = \frac{1}{4\pi}\left[x + \frac{\sin 2nx}{2n}\right]_0^{2\pi} = \frac{1}{2}.$$

Suppose the function $\varphi(x)$ is not normalized. Then, (if it does not vanish identically), it always can be transformed into a normalized function by multiplication by a suitable constant. In fact, the mean value

$$\frac{1}{b-a}\int_a^b |\varphi(x)|^2\,dx = M\{|\varphi|^2\}$$

is always real and in this case greater than zero. Consequently, the function

$$\frac{\varphi(x)}{\sqrt{M\{|\varphi|^2\}}}$$

has meaning and is obviously normalized. Thus, for example, if n denotes a positive integer, $\sqrt{2}\cos nx$ is a normalized function in $0 \leqq x \leqq 2\pi$ and more generally in any interval $a \leqq x \leqq a + 2\pi$.

13. A (finite or infinite) set of functions continuous in $a \leqq x \leqq b$

$$\varphi(x),\quad \psi(x),\quad \ldots$$

is called an orthogonal system, if every two of the func-

tions are orthogonal to each other. The set will be called normal, if each function of the system is normalized in the sense of the last article.

A normal orthogonal system is, therefore, a system of functions for which the relation

$$M\{\varphi\overline{\psi}\} = 0$$

holds for any two distinct functions of the system and for which the relation

$$M\{\varphi\overline{\varphi}\} = M\{|\varphi|^2\} = 1$$

holds for any single function of the system.

Example. The system of functions

$$e^{inx}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

is a normal orthogonal system in $0 \leqq x \leqq 2\pi$ or more generally in each interval $a \leqq x \leqq a + 2\pi$ of length 2π. In fact,

$$M\{e^{inx}e^{-imx}\} = 0 \text{ for } n \neq m$$

and

$$M\{e^{inx}e^{-inx}\} = M\{1\} = 1$$

hold for each n. Each sub-set of the system under consideration is obviously itself a normal orthogonal system. The system as a whole, however, is complete (vollstaendig), in the sense that it is not a sub-system of a larger normal orthogonal system. This important property will be analyzed later in some detail.

FOURIER CONSTANTS WITH RESPECT TO A NORMALIZED ORTHOGONAL SYSTEM. THEIR MINIMAL PROPERTY. BESSEL'S FORMULA. BESSEL'S INEQUALITY.

14. Let $\{\varphi(x)\}$ denote a normal orthogonal system in the interval $a \leqq x \leqq b$ and $F(x) = U(x) + iV(x)$ an arbitrary function continuous in $a \leqq x \leqq b$. We call the number

$$F_\varphi = M\{F(x)\overline{\varphi(x)}\} = \frac{1}{b-a}\int_a^b F(x)\overline{\varphi(x)}\,dx.$$

the Fourier constant of $F(x)$ with respect to a function $\varphi(x)$ of an orthogonal system. Now, choosing a certain finite number of functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$, of the system $\{\varphi(x)\}$ and forming the sum

$$S(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + \cdots + c_n\varphi_n(x).$$

where $c_1, c_2, \ldots, c_n$ are complex constants, we propose the following

Problem. The constants $c_1, c_2, \ldots, c_n$ are to be so chosen that the sum $S(x)$ approximates the given function $F(x)$ in the best possible manner in the sense of the method of least squares. In other words the mean value

$$M\{|F(x) - S(x)|^2\} = \frac{1}{b-a}\int_a^b |F(x) - S(x)|^2 dx$$

is to be as small as possible.

It will be shown that this problem has one and only one solution.

Through direct calculation and application of the identity $|A|^2 = A\bar{A}$), we obtain

$$M\{|F-S|^2\} = M\left\{\left(F(x) - \sum_{\nu=1}^{n} c_\nu \varphi_\nu(x)\right)\left(\overline{F(x)} - \sum_{\nu=1}^{n} \bar{c}_\nu \overline{\varphi_\nu(x)}\right)\right\}$$

$$= M\{F\bar{F}\} - \sum_{\nu=1}^{n} \bar{c}_\nu M\{F\overline{\varphi_\nu}\} - \sum_{\nu=1}^{n} c_\nu M\{\bar{F}\varphi_\nu\}$$

$$+ \sum_{\nu_1=1}^{n} \sum_{\nu_2=1}^{n} c_{\nu_1} \overline{c_{\nu_2}} M\{\varphi_{\nu_1} \overline{\varphi_{\nu_2}}\},$$

and thus, on account of

$$M\{\varphi_{\nu_1}\overline{\varphi_{\nu_2}}\} = \begin{cases} 0 \text{ for } \nu_1 \neq \nu_2 \\ 1 \text{ for } \nu_1 = \nu_2, \end{cases}$$

$$M\{|F-S|^2\} = M\{|F|^2\} - \sum_1^n \bar{c}_\nu F_{\varphi_\nu} - \sum_1^n c_\nu \overline{F_{\varphi_\nu}} + \sum_{\nu=1}^{n} c_\nu \bar{c}_\nu$$

$$= M\{|F|^2\} + \sum_1^n (c_\nu - F_{\varphi_\nu})(\bar{c}_\nu - \overline{F_{\varphi_\nu}}) - \sum_1^n F_{\varphi_\nu}\overline{F_{\varphi_\nu}}.$$

We thus obtain the simple formula

$$M\{|F(x) - S(x)|^2\} = M\{|F(x)|^2\} - \sum_1^n |F_{\varphi_\nu}|^2 + \sum_1^n |c_\nu - F_{\varphi_\nu}|^2.$$

This formula gives us the solution of the proposed problem, by showing us that to make the mean value

$$M\{|F(x) - S(x)|^2\}$$

as small as possible, we must choose the constants $c_1, c_2, \ldots,$ c_n as the Fourier constants $F_{\varphi_1}, F_{\varphi_2}, \ldots, F_{\varphi_n}$. In this case only, does the last term of the formula (the one in which the constants c_ν appear) assume its smallest possible value, namely, zero.

This minimal property shows the meaning of the Fourier constants. It should be noted that the solution of the prob-

lem is unique and that the value which must be given to each constant c_ν depends only on the corresponding function $\varphi_\nu(x)$ and not on any other functions of the system.

15. If in the above result the Fourier constant c_ν is substituted for the constant F_{φ_ν}, we obtain Bessel's formula

$$M\left\{|F(x)-\sum_1^n F_{\varphi_\nu}\varphi_\nu(x)|^2\right\} = M\{|F(x)|^2\} - \sum_1^n |F_{\varphi_\nu}|^2 .$$

Since the left hand member of this formula is obviously ≥ 0 we obtain Bessel's inequality as a corollary:

$$\sum_1^n |F_{\varphi_\nu}|^2 \leq M\{|F(x)|^2\} .$$

This is also valid for an arbitrary number of functions $\varphi_\nu(x)$ selected from our system.

FOURIER SERIES OF PERIODIC FUNCTIONS

16. We now proceed to make a closer study of the most important of all orthogonal systems, namely the previously mentioned normal orthogonal system

$$e^{inx}, \qquad n = 0, \pm 1, \pm 2 \ \ldots$$

in the interval $0 \leq x \leq 2\pi$.

We let $P(x)$ denote an arbitrary continuous function of period 2π. (Later on we shall use the notation $P_1(x), \ldots$). We represent the nth Fourier coefficient of $P(x)$ by

$$a_n = M\{P(x)\,e^{-inx}\} = \frac{1}{2\pi}\int_0^{2\pi} P(x)\,e^{-inx}\,dx ,$$

and we call the trigonometric series $\sum_{-\infty}^{\infty} a_n e^{inx}$ formed with the coefficients a_n , the Fourier series of $P(x)$. we use for this the abbreviation $P(x) \sim \sum a_n e^{inx}$.

We stress the fact that up to now we have been dealing with a purely formal matter. We can say nothing at present of the arrangement of the terms, or of the convergence of the series, etc.

17. To justify the selection of these numbers a_n as coefficients, we point out the following:

1. According to the minimal property (§ 14), every

finite sum $\sum^* a_n e^{inx}$ of terms of the Fourier series approximates the function $P(x)$ better in the mean than any other linear combination $\sum^* c_n e^{inx}$ of the same functions e^{inx}.
This means that

$$M\{|P(x) - \sum^* a_n e^{inx}|^2\} < M\{|P(x) - \sum^* c_n e^{inx}|^2\},$$

as long as it is not true that $c_n = a_n$ for all values of n involved.

From now on, the symbol $\sum^* \dots$ will always denote a finite sum.

2. If we make the formal substitution,

$$P(x) = \sum_{-\infty}^{\infty} b_n e^{inx}$$

then, after multiplying by e^{-inx} and forming mean values, we obtain simply

$$M\{P(x)\, e^{-inx}\} = b_n .$$

There is one important case where this formal type of calculation is clearly permissible, namely in the following

Special Theorem. If $P(x)$ can be developed into a trigonometric series $\sum b_n e^{inx}$, which converges uniformly for all x (regardless of the order of the terms) then this series must be none other than the Fourier series of $P(x)$.

In this case the series resulting from the multiplication by e^{-inx} also converges uniformly. The term by term integration is, therefore, permissible and it yields $a_n = b_n$ for each n.

With later applications in mind, we restate our theorem in the following form:

> If a trigonometric series $\sum b_n e^{inx}$ converges uniformly (regardless of the order of its terms) with $S(x)$, as its sum, then it is precisely the Fourier Series of the sum $S(x)$.

For example, a "trigonometric polynomial" $\sum^* b_n e^{inx}$ is the Fourier Series of its sum $S(x)$.

OPERATIONS WITH FOURIER SERIES

18. We are now going to examine the connection between the functions $P(x)$ and their Fourier series -- at first from a rather formal standpoint. We wish to show, for instance, that simple operations with functions $P(x)$ are paralleled by

the corresponding formal operations with their Fourier series.

From $P(x) \sim \sum a_n e^{inx}$, the simple formula

(1) $$kP(x) \sim \sum k a_n \cdot e^{inx},$$

is seen immediately to follow, where k denotes an arbitrary complex constant. In fact,

$$M\{kP(x)\, e^{-inx}\} = kM\{P(x)\, e^{-inx}\} = k a_n.$$

(2) $$e^{imx} P(x) \sim \sum a_n e^{i(m+n)x}, \text{ i.e., } \sum a_{n-m} e^{inx}.$$

For $$M\{e^{imx} P(x)\, e^{-inx}\} = M\{P(x)\, e^{-i(n-m)x}\} = a_{n-m}.$$

(3) $$P(x+k) \sim \sum a_n e^{in(x+k)}, \text{ i.e., } \sum a_n e^{ink} \cdot e^{inx},$$

where k denotes an arbitrary real constant. For

$$M\{P(x+k)\, e^{-inx}\} = e^{ink} M\{P(x+k)\, e^{-in(x+k)}\},$$

Hence, if we set $x + k = x'$

$$= e^{ink} M\{P(x')\, e^{-inx'}\} = e^{ink} \cdot a_n.$$

(4) $$\overline{P(x)} \sim \sum \overline{a_n}\, e^{-inx}, \text{ i.e., } \sum \overline{a_{-n}}\, e^{inx}$$

since $$M\{\overline{P(x)}\, e^{-inx}\} = \overline{M\{P(x)\, e^{inx}\}} = \overline{a_{-n}}.$$

From $P_1(x) \sim \sum a_n e^{inx}$ and $P_2(x) \sim \sum b_n e^{inx}$ it further follows that

(5) $$P_1(x) + P_2(x) \sim \sum (a_n + b_n)\, e^{inx}.$$

since

$$M\{(P_1(x) + P_2(x))\, e^{-inx}\} = M\{P_1(x)\, e^{-inx}\} + M\{P_2(x)\, e^{-inx}\} = a_n + b_n$$

(From the formulas (1) and (5) it follows, for example, that the Fourier series of the difference $P_1(x) - P_2(x)$ results from the formal subtraction of the Fourier series of $P_1(x)$ and $P_2(x)$.)

19. From two of the above formulas, namely (1) and (2), we arrive at the formula for the multiplication of an arbitrary periodic function $P(x)$ by a rather special function. The following <u>general multiplication theorem</u> is really deeper than the two preceding theorems:

(6) $$P_1(x) P_2(x) \sim \sum c_n e^{inx} \text{ with } c_n = \sum_{\mu+\nu=n} a_\mu b_\nu.$$

These formulas can first be proved, however, only later (in §§ 27-29.).

20. It is simple, and yet important for later applications, to find the Fourier series of a so called folded function (gefalteten Funktion) or convolution function

$$Q(x) = \underset{t}{M}\{P_1(x+t)\,P_2(t)\} = \frac{1}{2\pi}\int_0^{2\pi} P_1(x+t)\,P_2(t)\,dt.$$

This function is obviously another continuous periodic function of period 2π, and the formula

$$(7) \qquad Q(x) = \underset{t}{M}\{P_1(x+t)\,P_2(t)\} \sim \sum a_n b_{-n} e^{inx}$$

is valid. This follows immediately by a simple change of the order of integration. In fact such a procedure yields:

$$\begin{aligned}\underset{x}{M}\{Q(x)\,e^{-inx}\} &= \underset{x}{M}\{\underset{t}{M}\{P_1(x+t)\,P_2(t)\}\,e^{-inx}\} \\ &= \underset{t}{M}\{P_2(t)\,\underset{x}{M}\{P_1(x+t)\,e^{-inx}\}\} = \underset{t}{M}\{P_2(t)\,a_n e^{int}\} \\ &= a_n \underset{t}{M}\{P_2(t)\,e^{int}\} = a_n b_{-n}.\end{aligned}$$

A certain special case is rather important. For $P_1(x) = P(x) \sim \sum a_n e^{inx}$ and $P_2(x) = \overline{P(x)} \sim \sum \overline{a_{-n}} e^{inx}$ this formula gives:

$$(8) \quad Q(x) = \underset{t}{M}\{P(x+t)\,\overline{P(t)}\} \sim \sum a_n \overline{a_{+n}}\, e^{inx}, \text{ i.e., } \sum |a_n|^2 e^{inx}.$$

The two formulas (7) and (8) for setting up the Fourier series for the functions $F(t) = P_1(x+t)P_2(t)$ and $P(x+t)\,\overline{P(t)}$ respectively are particularly interesting for the reason that a formal derivation of the multiplication theorem is desired. We might have expected the Formulas (7) and (8) to be as deep as (6). Yet we have just seen that this is not the case. We shall later give a proof of the multiplication theorem just on the basis of formula (7).

21. If a given sequence of periodic functions

$$P_m(x) \sim \sum a_n^{(m)} e^{inx}$$

converges uniformly for all x towards a limit function $P(x)$ as $m \to \infty$ then

$$(9) \qquad P(x) = \lim_{m\to\infty} P_m(x) \sim \lim_{m\to\infty} \sum a_n^{(m)} e^{inx}.$$

By this we mean that if $P(x) \sim \sum a_n e^{inx}$, then for each n the following formula is valid

$$a_n = \lim_{m\to\infty} a_n^{(m)}$$

and indeed, uniformly with respect to n.

The proof follows immediately from the formula

$$a_n - a_n^{(m)} = M\{(P(x) - P_m(x))\, e^{-inx}\}.$$

If $\varepsilon > 0$ is prescribed arbitrarily, then there exists a $M = M(\varepsilon)$, such that as long as $m > M$,

$$|P(x) - P_m(x)| < \varepsilon$$

for all x. It follows from this, that

$$|a_n - a_n^{(m)}| < M\{\varepsilon \cdot 1\} = \varepsilon$$

for $m > M$ and all n.

22. Let $P(x) \sim \sum a_n e^{inx}$ and let

$$P_1(x) = \int P(x)\, dx$$

be an arbitrary indefinite integral of $P(x)$. For arbitrary x

$$P_1(x + 2\pi) - P_1(x) = \int_x^{x+2\pi} P(\xi)\, d\xi = 2\pi a_0,$$

hence $P_1(x)$ will be periodic with period 2π when and only when $a_0 = 0$. Here, the following formula is valid:

$$P_1(x) \sim C + \sum_{n \neq 0} \frac{a_n}{in} e^{inx}. \tag{10}$$

For, using integration by parts, we obtain the following result for $n \neq 0$

$$M\{P_1(x)\, e^{-inx}\} = \frac{1}{2\pi} \int_0^{2\pi} P_1(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \left[P_1(x) \frac{e^{-inx}}{-in} \right]_0^{2\pi} + \frac{1}{in} \cdot \frac{1}{2\pi} \int_0^{2\pi} P(x)\, e^{-inx}\, dx = 0 + \frac{a_n}{in}$$

TWO FUNDAMENTAL THEOREMS. THE UNIQUENESS THEOREM AND PARSEVAL'S EQUATION

23. From the preceding investigations on the relationship between a function $P(x)$ and its Fourier series, the fundamental question arises whether or not a function is uniquely determined by its Fourier series. That is, are the Fourier series corresponding to two distinct functions $P_1(x)$ and $P_2(x)$ distinct? The (affirmative) answer to this question will be given by the Uniqueness theorem given below. At this point

we shall content ourselves with another formulation of the theorem. Thus let us take

$$P_1(x) \sim \sum b_n e^{inx} \quad \text{and} \quad P_2(x) \sim \sum c_n e^{inx};$$

then as was previously noticed,

$$P(x) = P_1(x) - P_2(x) \sim \sum (b_n - c_n)\, e^{inx}.$$

The two functions $P_1(x)$ and $P_2(x)$ then have the same Fourier series when and only when all the Fourier coefficients of the function $P(x) = P_1(x) - P_2(x)$ are zero. Henceforth the uniqueness theorem can also be formulated as follows. There exists no non-vanishing function $P(x)$ for which all Fourier constants a_n are equal to zero. In other words, (since each non-vanishing function $P(x)$ can be normalized by multiplication by a suitable constant) there exists no function $P(x)$ which satisfies the following two conditions:

$$M\{P(x)\, e^{-inx}\} = 0, \qquad n = 0, \pm 1, \pm 2, \ldots,$$
$$M\{|P(x)|^2\} = 1.$$

In this form the uniqueness theorem states simply that the normal orthogonal system $\{e^{inx}\}$ can not be enlarged by the addition of a new periodic function of period 2π. Thus the system is complete.

24. Let $P(x) \sim \sum a_n e^{inx}$ again denote an arbitrary (continuous) periodic function of period 2π.

From Bessel's formula

$$M\{|P(x) - \sum\nolimits^* a_n e^{inx}|^2\} = M\{|P(x)|^2\} - \sum\nolimits^* |a_n|^2$$

(where $\sum^* \cdots$ denotes the sum of an arbitrary finite number of terms) the Bessel Inequality follows:

$$\sum\nolimits^* |a_n|^2 \leqq M\{|P(x)|^2\}$$

Accordingly, the infinite series (with non-negative terms)

$$\sum_{-\infty}^{\infty} |a_n|^2$$

must be convergent and in fact its sum must be

$$\leqq M\{|P(x)|^2\}.$$

(In particular $a_n \to 0$ as $|n| \to \infty$.] .) We write

$$D = M\{|P(x)|^2\} - \sum_{-\infty}^{\infty} |a_n|^2 \geqq 0.$$

The question of whether $D > 0$ or $D = 0$ is then of fundamental importance.

1. If $D = 0$, then for each $\varepsilon > 0$ we can choose a finite sum $\sum^* |a_n|^2$ of terms of the series $\sum_{-\infty}^{\infty} |a_n|^2$ so that

$$M\{|P(x)|^2\} - \sum^* |a_n|^2 < \varepsilon,$$

or

$$M\{|P(x) - \sum^* a_n e^{inx}|^2\} < \varepsilon,$$

and $P(x)$ can accordingly be approximated in the mean by trigonometric polynomials to any degree of accuracy.

2. If, however, $D > 0$, then for each trigonometric polynomial $\sum^* c_n e^{inx}$ with arbitrary coefficients, we have by the minimal property of Fourier constants

$$M\{|P(x) - \sum^* c_n e^{inx}|^2\} \geqq M\{|P(x) - \sum^* a_n e^{inx}|^2\};$$

and thus from Bessel's formula, we have the relation,

$$M\{|P(x) - \sum^* c_n e^{inx}|^2\} \geqq M\{|P(x)|^2\} - \sum^* |a_n|^2$$
$$\geqq M\{|P(x)|^2\} - \sum_{-\infty}^{\infty} |a_n|^2 = D > 0.$$

Now in this case $P(x)$ can not be approximated in the mean by trigonometric polynomials with arbitrary accuracy (in fact not with an error smaller than D).

We shall bring out in the proof that $D = 0$ always, i.e., that "Parseval's equation"

$$\sum_{-\infty}^{\infty} |a_n|^2 = M\{|P(x)|^2\}$$

holds for each function $P(x)$. In accordance with what has just been said this theorem is equivalent to the following; Every continuous function $P(x)$ of period 2π can be approximated in the mean as closely as desired by trigonometric polynomials $\sum^* c_n e^{inx}$.

25. Before we prove the two fundamental theorems, we wish to first discuss the relationship between them. To this end, we first prove:

1. The uniqueness theorem follows from Parseval's equation.

For this reason the Parseval equation is also frequently called the "completeness" relation.

Proof: Let $P(x)$ have all its Fourier constants a_n equal to zero; then it follows from the Parseval equation

$$M\{|P(x)|^2\} = \sum_{-\infty}^{\infty} |a_n|^2,$$

that $M\{|P(x)|^2\} = 0$, so that obviously $P(x)$ must vanish identically.

The proof of the converse is just as easy (but less trivial):

2. Parseval's equation follows from the uniqueness theorem.

Proof: Starting from $P(x) \sim \sum a_n e^{inx}$ we construct the following function (as in 20, formula (8)):

$$Q(x) = M_t\{P(x+t)\overline{P(t)}\} \sim \sum |a_n|^2 e^{inx}.$$

Since the series $\sum |a_n|^2$ is convergent (with sum $\leq M\{|P(x)|^2\}$), the last series $\sum |a_n|^2 e^{inx}$ (arranged in any order) converges uniformly for all x and is accordingly the Fourier series of its sum $S(x)$ The two functions $Q(x)$ and $S(x)$ have the same Fourier series (namely $\sum |a_n|^2 e^{inx}$), thus according to the uniqueness theorem $Q(x) = S(x)$, i.e.,

$$Q(x) = \sum |a_n|^2 e^{inx}.$$

If, in particular, we choose $x = 0$ this equation yields

$$Q(0) = M_t\{P(t)\overline{P(t)}\} = M\{|P(t)|^2\} = \sum |a_n|^2$$

Parseval's equation.

LEBESGUE'S PROOF OF THE UNIQUENESS THEOREM

26. To prove the two fundamental theorems, it suffices to establish the validity of either of them. We here give a proof of the uniqueness theorem due to Lebesgue.

For the later development of this book it would not have been necessary to give this proof; in fact the two fundamental theorems follow from the Féjer theorem proved below. Moreover, there is no uniqueness proof in the general theory of almost periodic functions that corresponds to the simple Lebesgue proof.

Let a continuous function $P(x)$ of period 2π have all its Fourier coefficients equal to zero, i.e., let

$$M\{P(x)\,e^{-inx}\} = 0$$

for all n. We are to show that $P(x)$ is identically zero.

It is no limitation of generality to take $P(x)$ real. In fact if all Fourier constants of $P(x)$ are zero the same is true for the conjugate function $\overline{P(x)}$ and hence for the two functions $\frac{1}{2}\left(P(x) + \overline{P(x)}\right)$ and $\frac{1}{2i}\left(P(x) - \overline{P(x)}\right)$, the real and imaginary parts of $P(x)$. These two functions can vanish however only when $P(x)$ is identically zero.

We will prove the theorem indirectly; we assume that $P(x)$ is not identically zero. Suppose, for instance, that $P(x_0) \neq 0$. It is no restriction of generality to take $x_0 = 0$ and $P(0) > 0$; otherwise we replace $P(x)$ by $P(x+x_0)$ or $-P(x+x_0)$. We now choose $d(<\pi)$ so small that $P(x) > c > 0$ for $-d < x < d$. We wish to construct a trigonometric polynomial $Q(x) = \sum^* c_n e^{inx}$ that takes only real values and that places special emphasis on the neighborhood of $x = 0$.

To this end we consider the function

$$\psi(x) = \cos x + (1 - \cos d)$$

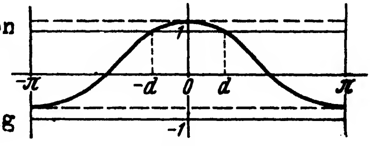

Fig. 1.

This function fulfills the following conditions.

1. $\psi(d) = \psi(-d) = 1$.
2. $|\psi(x)| < 1$ for $-\pi < x < -d$ and $d < x < \pi$.
3. $\psi(x) > 1$ for $-d < x < d$ and in particular

$$\psi(x) > g > 1 \text{ for } -\frac{d}{2} < x < \frac{d}{2}.$$

We now choose $Q(x) = (\psi(x))^N$, where N denotes any positive integer. Since $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, $(\psi(x))^N$ is a trigonometric

polynomial: $(\psi(x))^N = \sum^* c_n e^{inx}$. It follows, since $P(x)$ is by assumption orthogonal to all functions e^{inx} that $P(x)$ is also orthogonal to $(\psi(x))^N$ or, since both functions are real, that

$$\int_{-\pi}^{\pi} P(x)\,(\psi(x))^N\,dx = 0$$

For N sufficiently large, this leads to a contradiction. We denote the upper limit of $|P(x)|$ by K. Then, for every N

$$\int_{-\pi}^{\pi} P(x)\,(\psi(x))^N\,dx = \int_{-\pi}^{-d} + \int_{-d}^{-\frac{d}{2}} + \int_{-\frac{d}{2}}^{\frac{d}{2}} + \int_{\frac{d}{2}}^{d} + \int_{d}^{\pi} P(x)\,(\psi(x))^N\,dx$$

$$> -(\pi - d)\,K + 0 + dcg^N + 0 - (\pi - d)\,K > dcg^N - 2\pi K.$$

In this inequality, however, the right side $dcg^N - 2\pi K$ is surely >0, for N sufficiently large. (Indeed it approaches $+\infty$ as $N \to \infty$). This contradiction proves the uniqueness theorem.

THE MULTIPLICATION THEOREM

27. Now that we have proved the uniqueness theorem and Parseval's equation, we shall prove the general multiplication theorem (6) of § 19, and wish, indeed, to show that this theorem is equivalent to the two (mutually equivalent) principal theorems.

The general multiplication theorem states the following: If $P_1(x) \sim \sum a_n e^{inx}$ and $P_2(x) \sim \sum b_n e^{inx}$, then

$$(6) \qquad P_1(x)P_2(x) \sim \sum c_n e^{inx} \qquad c_n = \sum_{\mu+\nu=n} a_\mu b_\nu\,.$$

i.e., for each value of n

$$M\{P_1(x)\,P_2(x)\,e^{-inx}\} = \sum_{\mu+\nu=n} a_\mu b_\nu = \sum_{\mu=-\infty}^{\infty} a_\mu b_{n-\mu}\,.$$

Let us first notice that for each value of n, the series $\sum\limits_{\mu+\nu=n} a_\mu b_\nu = \sum\limits_{\mu=-\infty}^{\infty} a_\mu b_{n-\mu}$ is absolutely convergent. This follows immediately from the elementary inequality $|ab| \leq \frac{1}{2}(|a|^2 + |b|^2)$.

when we recall the convergence of the two series $\sum_{\mu=-\infty}^{\infty} |a_\mu|^2$ and $\sum_{\nu=-\infty}^{\infty} |b_\nu|^2 = \sum_{\mu=-\infty}^{\infty} |b_{n-\mu}|^2$.

We further notice that it suffices to consider the special case $n = 0$, so that we are now working with the constant term of the Fourier series, or with the equation

$$(*) \qquad M\{P_1(x)\, P_2(x)\} = \sum_{\mu+\nu=0} a_\mu b_\nu = \sum_\mu a_\mu b_{-\mu}.$$

For if this equation holds we can simply substitute the function $P_2(x)e^{-inx} \backsim \sum_\mu b_{n+\mu} e^{i\mu x}$ in place of $P_2(x)$, and obtain the more general equation

$$M\{P_1(x)\, P_2(x)\, e^{-inx}\} = \sum_{\mu=-\infty}^{\infty} a_\mu b_{n-\mu}.$$

28. We now proceed to the proof itself.

1. Parseval's equation (and therefore the uniqueness theorem also) follows from the multiplication theorem.

We choose, in particular, $P_2(x) = \overline{P_1(x)} \backsim \sum \overline{a_{-n}} e^{inx}$, then (*) yields the equation

$$M\{P_1(x)\, \overline{P_1(x)}\} = \sum_\mu a_\mu \overline{a_{+\mu}},$$

which is precisely the Parseval equation

$$M\{|P_1(x)|^2\} = \sum_\mu |a_\mu|^2.$$

2. The Multiplication theorem follows from uniqueness theorem (and thus from Parseval's equation).

The proof is completely analogous to an earlier proof. According to formula (7) in §20 we construct the function

$$Q(x) = M_t\{P_1(x+t)\, P_2(t)\} \backsim \sum_\mu a_\mu b_{-\mu} e^{i\mu x}.$$

Since (as we noticed before) the series $\sum_\mu a_\mu b_{-\mu}$ is absolutely convergent, the series $\sum_\mu a_\mu b_{-\mu} e^{i\mu x}$ (in any order of arrangement of terms) is uniformly convergent and is, therefore, the Fourier series of its sum $S(x)$. On the other hand, it is the Fourier series of $Q(x)$. Therefore, it follows from the uniqueness theorem that $Q(x) = S(x)$, or that

$$M_t\{P_1(x+t)\, P_2(t)\} = \sum_u a_\mu b_{-\mu} e^{i\mu x}.$$

If we choose $x = 0$, the desired relation (*) follows.

29. By the aid of a trick we shall derive the multiplication theorem directly from the more specialized Parseval equality without using the uniqueness theorem as an intermediary.

We have only to use the elementary identity

$$uv = \tfrac{1}{4}\{|u+\bar v|^2 - |u-\bar v|^2 + i|u+i\bar v|^2 - i|u-i\bar v|^2\}$$

If we apply this to the product $P_1(x)P_2(x)$, we get

$$M\{P_1P_2\}$$
$$= \tfrac{1}{4}[M\{|P_1+\overline{P}_2|^2\} - M\{|P_1-\overline{P}_2|^2\} + iM\{|P_1+i\overline{P}_2|^2\} - iM\{|P_1-i\overline{P}_2|^2\}].$$

If, further, we apply Parseval's equation to the functions $P_1(x)+\overline{P_2(x)}$, $P_1(x)-\overline{P_2(x)}$, $P_1(x)+i\overline{P_2(x)}$ and $P_1(x)-i\overline{P_2(x)}$ whose coefficients we know from formulas (1), (4) and (5) of § 18, we obtain the result that in the preceding relation the right hand member

$$= \tfrac{1}{4}\left[\sum_{-\infty}^{\infty}|a_n+\overline{b_{-n}}|^2 - \sum_{-\infty}^{\infty}|a_n-\overline{b_{-n}}|^2 + i\sum_{-\infty}^{\infty}|a_n+i\overline{b_{-n}}|^2 - i\sum_{-\infty}^{\infty}|a_n-i\overline{b_{-n}}|^2\right]$$

Now we can apply the identity again, but in the opposite direction, and obtain

$$M\{P_1P_2\} = \sum_{-\infty}^{\infty} a_n b_{-n}\,.$$

This is the desired relation (*).

SUMMABILITY OF FOURIER SERIES
FEJER'S THEOREM

30. We have seen that the Fourier Series $\sum a_n e^{inx}$ of a function $P(x)$ uniquely determines this function. This fact is related to the question of whether or not one can actually compute the function corresponding to a Fourier Series. To deal with this problem it will first be necessary to choose a particular ordering of the terms. (This was unnecessary for the preceding developments). We shall choose the "natural" order

$$a_0 + (a_1e^{ix} + a_{-1}e^{-ix}) + \cdots + (a_ne^{inx} + a_{-n}e^{-inx}) + \cdots.$$

As the nth, partial sum of the series, we take the sum

$$s_n(x) = \sum_{-n}^{n} a_\nu e^{i\nu x}.$$

We might next consider convergence. This would, however, be unwise; for as Du Bois Reymond first showed, there are

functions whose Fourier series (with terms taken in the order just given) are divergent at certain points. On the other hand we shall obtain excellent results if, with Féjer, we consider, not convergence, but summability.

31. We recall the definition of summability. (By summability we mean merely Cesaro, summability of the first order). Let $\sum_0^\infty u_n$ be a given infinite series and let $s_n = \sum_{\nu=0}^{n} u_\nu$ be the nth partial sum of the series. If the mean value is formed

$$S_n = \frac{1}{n}(s_0 + s_1 + \cdots + s_{n-1}) = \frac{1}{n}\sum_{\nu=0}^{n}(n-\nu)\,u_\nu = \sum_{\nu=0}^{n}\left(1-\frac{\nu}{n}\right)u_\nu$$

then the given sum is said to be summable with sum U, if $S_n \to U$ for $n \to \infty$.

Note: Summability is a more general conception than convergence, i.e., if the series $\sum_0^\infty u_n$ is convergent with sum U, then it is also summable with the same sum U. On the other hand there exist series that are summable but not convergent.

32. The Féjer theorem states the following:

The Fourier series

$$\sum_0^\infty u_n = a_0 + (a_1 e^{ix} + a_{-1} e^{-ix}) + \cdots + (a_n e^{inx} + a_{-n} e^{-inx}) + \cdots$$

of a continuous function $P(x)$ of period 2π is always summable and, moreover, uniformly summable, with sum $P(x)$, i.e., the mean value

$$S_n(x) = \frac{1}{n}(s_0(x) + s_1(x) + \cdots + s_{n-1}(x)) = \frac{1}{n}\sum_{\nu=-n}^{n}(n-|\nu|)\,a_\nu e^{i\nu x}$$

$$= \sum_{\nu=-n}^{n}\left(1-\frac{|\nu|}{n}\right)a_\nu e^{i\nu x}$$

approaches $P(x)$ uniformly for all x as $n \to \infty$.

The proof rests on the transformation of the expression for the mean value $S_n(x)$. Since for a fixed value of x

$$P(x+t) \sim \sum a_n e^{inx} \cdot e^{int},$$

we have first for the nth partial sum

$$s_n(x) = \sum_{\nu=-n}^{n} a_\nu e^{i\nu x} = \sum_{\nu=-n}^{n} \underset{t}{M}\{P(x+t)\, e^{-i\nu t}\}$$

$$= \underset{t}{M}\left\{P(x+t) \sum_{\nu=-n}^{n} e^{-i\nu t}\right\} = \underset{t}{M}\{P(x+t)\, D_n(t)\},$$

where the Dirichlet kernel $D_n(t)$ is given by

$$D_n(t) = \sum_{\nu=-n}^{n} e^{-i\nu t} = \frac{e^{i(n+\frac{1}{2})t} - e^{-i(n+\frac{1}{2})t}}{e^{i\frac{t}{2}} - e^{-i\frac{t}{2}}} = \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}} = \frac{\cos nt - \cos(n+1)t}{2\sin^2\frac{t}{2}}$$

For the mean value $S_n(x)$, we find a corresponding expression

$$S_n(x) = \sum_{\nu=-n}^{n} \left(1 - \frac{|\nu|}{n}\right) a_\nu e^{i\nu x} = \underset{t}{M}\{P(x+t)\, K_n(t)\},$$

where $K_n(t)$ is the Féjer kernel given by

$$K_n(t) = \sum_{\nu=-n}^{n} \left(1 - \frac{|\nu|}{n}\right) e^{-i\nu t} = \frac{1}{n} \sum_{\nu=-n}^{n} (n - |\nu|)\, e^{-i\nu t}$$

$$= \frac{1}{n}(D_0(t) + D_1(t) + \cdots + D_{n-1}(t))$$

$$= \frac{1}{n} \frac{1 - \cos nt}{2\sin^2\frac{t}{2}} = \frac{1}{n} \left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2$$

Of these expressions for $K_n(t)$ the first and (particularly) the last are important. From the first it follows that

$$M\{K_n(t)\} = 1$$

(since the constant term is 1) and from the last that — unlike $D_n(t)$ —

$$K_n(t) \geqq 0$$

for all t. (If t is a multiple of 2π, then $K_n(t) = n$; if however, nt, but not t, is a multiple of 2π, then $K_n(t) = 0$).

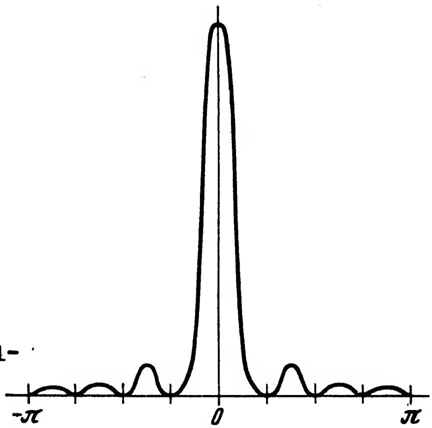

Fig. 2.

Now let $\varepsilon > 0$ be assigned an arbitrary value. We have to show that

$$|S_n(x) - P(x)| < \varepsilon$$

for all x, when $n > N = N(\varepsilon)$. For an arbitrary fixed x

$$S_n(x) - P(x) = \underset{t}{M}\{P(x+t)K_n(t)\} - P(x)\underset{t}{M}\{K_n(t)\}$$
$$= \underset{t}{M}\{(P(x+t) - P(x))K_n(t)\}.$$

Now $P(x)$ is continuous and periodic, hence $P(x)$ is bounded and uniformly continuous, i.e.,

$$|P(x)| \leqq C$$

for all x, and $|P(x_1) - P(x_2)| \leqq \eta$ for $|x_1 - x_2| \leqq \delta = \delta(\eta)$. We denote by $\omega(\delta)$, the upper limit of $|P(x_1) - P(x_2)|$ for $|x_1 - x_2| \leqq \delta$, and then $\omega(\delta) \to 0$ for $\delta \to 0$. Since we make use of the specified properties of $K_n(t)$ we obtain now for an arbitrary δ $(< \pi)$

$$|S_n(x) - P(x)| \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi}|P(x+t) - P(x)|K_n(t)\,dt$$

$$= \frac{1}{2\pi}\left[\int_{-\delta}^{\delta} + \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right]|P(x+t) - P(x)|K_n(t)\,dt$$

$$\leqq \frac{1}{2\pi}\left[\omega(\delta)\int_{-\delta}^{\delta}K_n(t)\,dt + 2(\pi - \delta)2C\cdot\frac{1}{n}\,\frac{1}{\sin^2\frac{\delta}{2}}\right]$$

$$< \frac{1}{2\pi}\left[\omega(\delta)\int_{-\pi}^{\pi}K_n(t)\,dt + 2\pi\frac{1}{n}\,\frac{2C}{\sin^2\frac{\delta}{2}}\right]$$

$$= \omega(\delta) + \frac{1}{n}\,\frac{2C}{\sin^2\frac{\delta}{2}}.$$

We first choose a <u>fixed</u> δ so small that $\omega(\delta) < \varepsilon/2$, and then an N so large that $\frac{1}{N}\,\frac{2C}{\sin^2\frac{\delta}{2}} < \frac{\varepsilon}{2}$. Then for $n > N$ (and all x)

$$|S_n(x) - P(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which proves the theorem.

33. The point should be brought out that the proof of Fejer's theorem was given independently of the principal theorems preceding it. The corresponding theorems in the theory of almost periodic functions have a different inter-

relation. It is, therefore, of interest to note this independence since, on the other hand, the two principal theorems can be shown to follow from Féjer's theorem.

Fejer's theorem is seen to contain the uniqueness theorem on the basis of the fact that the former theorem gives an algorithm for arriving at the function $P(x)$ from the Fourier series of $P(x)$

It will now be seen that Féjer's theorem also contains Parseval's equation.

WEIERSTRASS' THEOREM

34. Fejer's theorem contains as a special case a classical theorem of Weierstrass which can be considered to be the true fundamental theorem of the theory of (continuous) periodic functions.

Every continuous function $P(x)$ of period 2π can be approximated uniformly for all x by trigonometric polynomials $\sum^* c_n e^{inx}$, i.e., for every given $\varepsilon > 0$ a sum $\sum^* c_n e^{inx}$ can be given such that

$$|P(x) - \sum^* c_n e^{inx}| < \varepsilon$$

for all x.

This theorem is contained in Fejer's theorem and it can be deduced immediately from it if one disregards the special nature of the trigonometric sum $S_n(x)$. Weierstrass' theorem states less than Fejer's theorem; for example, it follows immediately from the latter that in the approximating sum $\sum^* c_n e^{inx}$ we need use only those harmonics e^{inx} which actually appear in the Fourier series of $P(x)$, i.e., whose Fourier coefficients a_n are distinct from zero. The Weierstrass theorem, however, contains the Parseval's equality, which, indeed, asserts only the possibility of approximation of $P(x)$ by trigonometric polynomials in the mean. In order to have

$$M\{|P(x) - \sum^* c_n e^{inx}|^2\} < \varepsilon$$

it suffices to have

$$|P(x) - \sum^* c_n e^{inx}| < \sqrt{\varepsilon}$$

for all x.

35. For later use it is convenient to have the Weierstrass theorem stated in a slightly different form.

We consider the set of finite sums of the form

$$S(x) = \sum{}^{*} c_n e^{inx}$$

(with arbitrary complex coefficients) and extend this set by adding to it all functions $f(x)$ which can be approximated uniformly by such sums $S(x)$. This extended class of functions will be called the closure (abgeschlossene Huelle) of $\{S(x)\}$ and will be denoted by $H\{S(x)\}$.

Then this set of functions $H\{S(x)\}$ is identical with the class of all continuous periodic functions $P(x)$ of period 2π.

For on one hand, every function belonging to $H\{S(x)\}$ is continuous and periodic with period 2π since each function $S(x)$, has these properties and so must every function which is the limit of a uniformly convergent sequence of functions $S(x)$. On the other hand, every continuous periodic function $P(x)$ of period 2π belongs to $H\{S(x)\}$, for this is precisely the Weierstrass theorem.

36. With this we conclude our study of the theory of continuous periodic functions. It should be noticed that one can prove Weierstrass' theorem without recourse to Fourier series, and that this theorem then provides perhaps the simplest approach to the fundamental theorems of Fourier series. Our particular presentation was chosen only because it provides a basis for the development of the theory of almost periodic functions. In fact this method is the natural (and until now the usual) method, since the theory of almost periodic functions can be discussed simply in terms of approximation by sums $\sum{}^{*} c_n e^{i\lambda_n x}$ with arbitrary real exponents λ_n.

TWO REMARKS

37. We have hitherto considered only continuous Periodic functions $P(x)$ in anticipation of Weierstrass' theorem which holds only for continuous functions. The principal theorems of Fourier theory retain their validity for more

general classes of functions. One such result will be necessary for a later application to the theory of almost periodic functions.

Let $P(x)$ be periodic with period 2π and continuous everywhere with the exception of the points $0, \pm 2\pi, \pm 4\pi, \ldots$ at each of which $P(x)$ has a saltus or "jump" (i.e., where

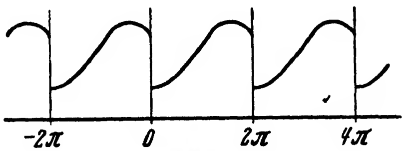

Fig. 3.

right and left hand limits exist but are distinct). Further let

$$\sum_{-\infty}^{\infty} a_n e^{inx} \quad \text{with} \quad a_n = \frac{1}{2\pi}\int_0^{2\pi} P(x)\, e^{-inx}\, dx$$

be the Fourier series of $P(x)$. We wish to show, that under these circumstances Parseval's equation

$$(*) \qquad \frac{1}{2\pi}\int_0^{2\pi} |P(x)|^2\, dx = \sum_{-\infty}^{\infty} |a_n|^2$$

remains valid; and, moreover, the following, somewhat stronger, equation holds:

$$\frac{1}{2\pi}\int_0^{2\pi} P(x+t)\, \overline{P(t)}\, dt = \sum_{-\infty}^{\infty} |a_n|^2 e^{inx}$$

Proof: Exactly as in § 24, one can show that the series $\sum_{-\infty}^{\infty} |a_n|^2$, converges (moreover, with a sum $\leq M\{|P(x)|^2\}$), and accordingly the series $\sum_{-\infty}^{\infty} |a_n|^2 e^{inx}$, also converges, and in fact uniformly in x. It also follows, exactly as in § 20, that the folded function

$$Q(x) = \frac{1}{2\pi}\int_0^{2\pi} P(x+t)\, \overline{P(t)}\, dt$$

(which is also periodic of period 2π) has as its Fourier series exactly the series $\sum_{-\infty}^{\infty} |a_n|^2 e^{inx}$. Now, however, the function $Q(x)$, (despite the discontinuity of the integrand $P(x+t)\,\overline{P(t)}$)) is an everywhere continuous function of x .

Therefore, we can apply immediately the uniqueness theorem for continuous functions and get the desired equation

$$Q(x) = \sum_{-\infty}^{\infty} |a_n|^2 e^{inx},$$

which becomes Parseval's equation (*) for $x = 0$.

38. Finally we note that the simple change of variable

$$\frac{2\pi}{T} x = x'$$

makes it possible to extend our results immediately to periodic functions whose period is not 2π, but an arbitrary positive number T. Let $F(x)$ be such a function, which may have jumps at the points $mT\,(m = 0, \pm 1, \pm 2, \ldots)$ then the Fourier series will be of the form

$$F(x) \sim \sum_{-\infty}^{\infty} a_n e^{in\frac{2\pi}{T}x}$$

where

$$a_n = M\left\{F(x)\, e^{-in\frac{2\pi}{T}x}\right\} = \frac{1}{T}\int_0^T F(x)\, e^{-in\frac{2\pi}{T}x}\, dx,$$

and likewise Parseval's Equation

$$\frac{1}{T}\int_0^T |F(x)|^2\, dx = \sum_{-\infty}^{\infty} |a_n|^2$$

is valid as well as its generalization

$$G(x) = \frac{1}{T}\int_0^T F(x+t)\,\overline{F(t)}\, dt = \sum_{-\infty}^{\infty} |a_n|^2 e^{in\frac{2\pi}{T}x}.$$

THE THEORY OF
ALMOST PERIODIC FUNCTIONS

THE MAIN PROBLEM OF THE THEORY

39. We consider the set of all finite sums

$$s(x) = \sum_{n=1}^{N} a_n e^{i\lambda_n x}, \qquad -\infty < x < \infty,$$

where the coefficients a_n are arbitrary complex quantities while the exponents λ_n are arbitrary real quantities. The set of these functions $s(x)$ will now be "closed" by the addition of those functions $f(x) = u(x) + iv(x)$ which can be approximated by such sums $s(x)$ uniformly for all x. Thus for any $\varepsilon > 0$, some $s(x)$ exists satisfying

$$|f(x) - s(x)| \leqq \varepsilon \quad \text{for} \quad -\infty < x < \infty;$$

(These functions $f(x)$ are obviously continuous functions of x). We call the function class obtained by this extension the closure of the set $\{s(x)\}$ and we denote it by $H\{s(x)\}$.

Thus the main problem of the theory consists in characterizing the functions $f(x)$ of this class $H\{s(x)\}$ by "structural properties" bearing no direct relationship to the concepts used in the definition of $H\{s(x)\}$.

Remarks: A fundamental difference exists between this problem and the corresponding one (solved by Weierstrass) for the case of harmonic sums $\sum^* a_n e^{inx}$. For in the former case the exponents λ_n are chosen from the non-enumerable set of real numbers $-\infty < \lambda < \infty$ and not just from the enumerable set $\lambda = n (= 0, \pm 1, \pm 2, \ldots)$. Later we shall see that the functions $f(x)$ corresponding to the class $H\{s(x)\}$ help us overcome this difficulty by permiting us to pick out distinct enumerable sets of exponents $\lambda_1, \lambda_2, \lambda_3, \ldots$ which take the place of the set of exponents $0, \pm 1, \pm 2, \ldots$.

TRANSLATION NUMBERS

40. For a beginning we consider at first a simple example of a function of the class $H\{s(x)\}$, namely the function

$$s(x) = e^{i\lambda_1 x} + e^{i\lambda_2 x},$$

where λ_1 and λ_2 denote real numbers distinct from zero. the two terms $e^{i\lambda_1 x}$ and $e^{i\lambda_2 x}$ individually are each periodic with the periods respectively of $p_1 = 2\pi/|\lambda_1|$ and $p_2 = 2\pi/|\lambda_2|$ At the same time, of course, all their integral multiples are also periods of $e^{i\lambda_1 x}$ and $e^{i\lambda_2 x}$. There are two cases:

1. λ_1/λ_2 is rational, i.e., p_1/p_2 is rational or $n_1 p_1 = n_2 p_2$ for two integers n_1 and n_2 distinct from zero. In this trivial case the two terms $e^{i\lambda_1 x}$ and $e^{i\lambda_2 x}$ have the common period $P = n_1 p_1 = n_2 p_2$, and their sum $s(x)$ is purely periodic with this period P.

2. λ_1/λ_2 is irrational, i.e., p_1/p_2 is irrational. In this case no multiple of p_1 equals a multiple of p_2 i.e. the two arithmetic progressions $(0, \pm p_1, \pm 2p_1, \ldots)$ and $(0, \pm p_2, \pm 2p_2, \ldots)$ have the origin as their only common point. In this case the function $s(x)$ is <u>not</u> periodic: e.g., it has the value 2 for $x = 0$ and for no other value of x.

On the other hand, as can easily be shown, there exists for any given $\delta > 0$ a pair of arbitrarily great integers n_1 and n_2, such that

$$|n_1 p_1 - n_2 p_2| < \delta$$

(This is a special case of a general theorem in Diophantine Approximation). Let τ be a number which is "near" to $n_1 p_1$ and $n_2 p_2$, (for example, a number between $n_1 p_1$ and $n_2 p_2$) then τ is "almost" a period of $e^{i\lambda_1 x}$ as well as $e^{i\lambda_2 x}$ and accordingly "almost" a period of its sum $s(x)$, the difference $s(x+\tau) - s(x)$ is very small for all x.

41. From these considerations we shall be led to the formal definition of a "translation number (Verschiebungszahl)". Let us take an arbitrary function $f(x) = u(x) + iv(x)$ continuous for $-\infty < x < \infty$. The real number τ will then be called a <u>translation number of $f(x)$ corresponding to</u> ε (and denoted by $\tau(\varepsilon)$ or $\tau_f(\varepsilon)$) whenever

$$|f(x+\tau) - f(x)| \leqq \varepsilon \qquad \text{for} \qquad -\infty < x < \infty.$$

<u>Remarks</u>: Obviously a translation number of $f(x)$ corresponding to ε corresponds a fortiori to every quantity $\varepsilon_1 > \varepsilon$ and together with τ, $-\tau$ is also a translation number of $f(x)$ corresponding to ε. It is further true that $\tau(\varepsilon_1) + \tau(\varepsilon_2) = \tau(\varepsilon_1 + \varepsilon_2)$ and $\tau(\varepsilon_1) - \tau(\varepsilon_2) = \tau(\varepsilon_1 + \varepsilon_2)$) i.e., the sum or difference of translation numbers corresponding to ε_1 or

ε_2 , will at the same time be a translation number corresponding to $\varepsilon_1 + \varepsilon_2$.

DEFINITION OF ALMOST PERIODICITY

42. After the special case studied above, it is natural to consider the class of functions $f(x)$ which have the following property: For every $\varepsilon > 0$ there exists a translation number $\tau = \tau(\varepsilon)$ of $f(x)$ and these numbers $\tau(\varepsilon)$ are arbitrarily great. This is not, however, a happy choice. Not only is the class of functions $f(x)$ far from the desired class $H\{s(x)\}$, but it can even be shown, by constructing suitable examples, that this class does not even remain invariant under the ordinary operation of addition, i.e., in case $f(x)$ and $g(x)$ are two functions of this class their sum $f(x) + g(x)$ need not belong to this class.

43. To be able to conclude that $f(x)$ belongs to $H\{s(x)\}$ more must be said about $f(x)$ than merely that it has arbitrarily large translation numbers for each $\varepsilon > 0$. To this end, we introduce the concept of "relative density". A set E of real numbers τ will be called <u>relatively dense</u> (<u>relativ dicht</u>) if there are no arbitrarily large gaps among the numbers τ or, to be exact, if some length L exists such that every interval $\alpha < x < \alpha + L$ of this length contains at least one number τ of the set E.

<u>Example</u>: The numbers np $(n = 0, \pm 1, \pm 2, \ldots, p > 0)$ of an arithmetic progression are relatively dense and indeed the simplest such set possible. The set of numbers $\pm n^2$ $(n = 1, 2, \ldots)$, on the contrary, is not relatively dense (since $(n+1)^2 - n^2 = 2n + 1 \to \infty$ as $n \to \infty$). Roughly speaking, a "relatively dense" set can be described as one which is "just as dense" as an arithmetic progression.

44. We now come to the real definition of an "almost periodic" function.

MAIN DEFINITION. <u>A function continuous for $-\infty < x < \infty$ will be called almost periodic when, given an $\varepsilon > 0$, there exists a relatively dense set of translation numbers of $f(x)$ corresponding to ε. In other words, to every $\varepsilon > 0$, a</u> length $L = L(\varepsilon)$ of some sort exists such that each interval of length $L(\varepsilon)$ contains at least one translation number

$\tau = \tau(\varepsilon)$.

Examples: A continuous purely periodic function of period p is obviously a special case of an almost periodic function. In fact here we can take the periods np ($n = 0, \pm 1, \pm 2, \ldots$) as translation numbers $\tau(\varepsilon)$ for each ε. It is further seen that if $f(x)$ is almost periodic, the same is true of $f(x+c)$ (c an arbitrary real constant) of $cf(x)$ (c an arbitrary complex constant) of $\overline{f(x)}$ and also of $|f(x)|$ since $\big||f(x+\tau)| - |f(x)|\big| \leqq |f(x+\tau) - f(x)|$).

In case $f(x)$ is not purely periodic every usable length $l(\varepsilon)$ must be arbitrarily large if ε is chosen small enough. Otherwise it would be possible to use one and the same length for every ε. For example, the interval $L \leqq x \leqq 2L$ would contain a point of accumulation of arbitrarily fine translation numbers τ, (i.e., translation numbers corresponding to arbitrarily small values of ε). Such a point of accumulation, contrary to assumption, would be a perio of $f(x)$.

The fundamental theorem of the theory of almost periodic functions can be demonstrated only at the end of this entire chapter. It states, that the class of almost periodic functions is identical with the class $H\{s(x)\}$. Thus almost periodicity is precisely the desired structural property which characterizes the functions $f(x)$ corresponding to $H\{s(x)\}$

TWO SIMPLE PROPERTIES OF ALMOST PERIODIC FUNCTIONS

45. We wish to develop the theory of almost periodic functions in direct analogy with the theory of purely periodic functions. Here we must note, however, that many theorems which are decidedly trivial for purely periodic functions are no longer trivial for almost periodic functions. This is illustrated by two theorems that we shall presently prove, according to which every almost periodic function is bounded and uniformly continuous. The corresponding theorems for purely periodic functions were trivial because in the investigation of such functions we observed that we could restrict ourselves to a finite interval. For almost periodic functions the proof -- a very simple one -- rests on a similar observation which is made possible by the

existence of a length $L = L(\varepsilon)$ as provided in the definition of almost periodicity.

From the property of the number $L = L(\varepsilon)$ it follows directly that if we consider a fixed interval J of length $L = L(\varepsilon)$ (e.g., the interval $0 < x < L$) we are always able to find a translation number $\tau = \tau(\varepsilon)$ $(= \tau(\varepsilon, x_0))$ for any x_0 whatsoever, such that a translation τ changes x_0 into some point $x_1 = x_0 + \tau$ of this interval J. (We have only to choose the

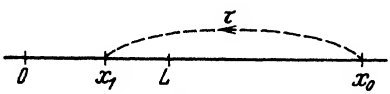

Fig. 4.

translation number so that it belongs to the interval $-x_0 < x < -x_0 + L$). Thus even in the case of almost periodic functions, this consideration makes it possible to restrict the work to a fixed, finite interval.

46. We now prove the above mentioned theorems.

THEOREM I. An almost periodic function is bounded i.e., there exists a constant $C = C(f)$, such that

$$|f(x)| \leqq C \quad \text{for} \quad -\infty < x < \infty .$$

Proof: First we specify the length $L = L(1)$ for $\varepsilon = 1$. In the closed interval $0 \leqq x \leqq L(1)$ the function $f(x)$ is bounded, say $|f(x)| \leqq c$. Then at each point x_0 the inequality $|f(x_0)| \leqq c + 1 = C$ will hold. In fact we could find a translation number $\tau = \tau(1)$ for an arbitrarily prescribed x_0 such that $0 < x_0 + \tau < L(1)$ and we thus obtain

$$|f(x_0)| \leqq |f(x_0 + \tau)| + |f(x_0) - f(x_0 + \tau)| \leqq c + 1 .$$

Corollary: If $f(x)$ is almost periodic, so is $(f(x))^2$. In fact we have

$$|(f(x+\tau))^2 - (f(x))^2| = |f(x+\tau) + f(x)| \cdot |f(x+\tau) - f(x)| \leqq 2C|f(x+\tau) - f(x)|,$$

and thus every translation number of $f(x)$ corresponding to $\varepsilon/2C$ is likewise a translation number of $(f(x))^2$ corresponding to ε.

In particular $|f(x)|^2$ is an almost periodic function.

As a simple application of Theorem I, let us take the following: If the almost periodic function $f(x) \neq 0$ for all x, then in order that $1/f(x)$ also be almost periodic it is nec-

essary that the lower limit γ of $|f(x)|$ for $-\infty < x < \infty$ be greater than zero. (This condition is not satisfied automatically). The condition $\gamma > 0$ is, however, sufficient; for according to an inequality valid for all τ, namely,

$$\left|\frac{1}{f(x+\tau)} - \frac{1}{f(x)}\right| = \left|\frac{f(x+\tau)-f(x)}{f(x)f(x+\tau)}\right| \leqq \frac{1}{\gamma^2}|f(x+\tau)-f(x)|$$

we find that the translation number of $f(x)$ corresponding to $\gamma^2\varepsilon$ is simultaneously a translation number of $1/f(x)$ corresponding to ε.

THEOREM II: <u>An almost periodic function $f(x)$ is uniformly continuous for $-\infty < x < \infty$</u> (and not merely for any finite interval) i.e., to an arbitrary $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ of such nature that

$$|f(x_1) - f(x_2)| \leqq \varepsilon \quad \text{for} \quad |x_1 - x_2| \leqq \delta.$$

<u>Proof</u>: ε being given, we first specify the length $L = L\left(\frac{\varepsilon}{3}\right)$ and consider the finite closed interval $-1 \leqq x \leqq L\left(\frac{\varepsilon}{3}\right) + 1$. We then determine a $\delta(<1)$, so that the inequality $|f(y_1) - f(y_2)| \leqq \frac{\varepsilon}{3}$ is valid for every pair of points (y_1, y_2) in the interval $\left(-1, L\left(\frac{\varepsilon}{3}\right) + 1\right)$ provided $|y_1 - y_2| \leqq \delta$. Then this δ will have the desired property. In fact let (x_1, x_2) be an arbitrary pair of points with $|x_1 - x_2| \leqq \delta$, and let the translation number $\tau = \tau\left(\frac{\varepsilon}{3}\right)$ be chosen so that the point $y_1 = x_1 + \tau$ lies in the interval $0 < x < L\left(\frac{\varepsilon}{3}\right)$ Then the point $y_2 = x_2 + \tau$ obviously lies in the interval $\left(-1, L\left(\frac{\varepsilon}{3}\right) + 1\right)$ and thus we obtain the equation

$$|f(x_1) - f(x_2)| \leqq |f(x_1) - f(y_1)| + |f(y_1) - f(y_2)| + |f(y_2) - f(x_2)|$$
$$\leqq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

<u>Remarks</u>: Theorem II is equivalent to the following statement. To every ε there exists a δ, so that <u>every</u> number τ satisfying the condition $|\tau| \leqq \delta$ yields a translation number corresponding to ε.

47. From Theorem II we can deduce a corollary which we will use often.

<u>Corollary</u>: For each ε there exists an L <u>and</u> a δ, such

that each interval $\alpha < x < \alpha + L$ of length L contains not just one translation number $\tau(\varepsilon)$, but rather an entire interval of length δ whose points are all translation numbers $\tau(\varepsilon)$.

Proof: Let $L\left(\frac{\varepsilon}{2}\right)$ and $\delta\left(\frac{\varepsilon}{2}\right)$ respectively, be chosen according to the definition of almost-periodicity and Theorem II. Then the two numbers $L = L\left(\frac{\varepsilon}{2}\right) + 2\delta\left(\frac{\varepsilon}{2}\right)$ and $\delta = 2\delta\left(\frac{\varepsilon}{2}\right)$ fulfill the conditions of this corollary. In fact a translation number $\tau\left(\frac{\varepsilon}{2}\right)$, lies in every interval of length $L\left(\frac{\varepsilon}{2}\right)$

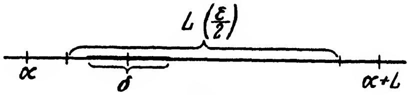

Fig. 5.

and further, each number of the form $\tau\left(\frac{\varepsilon}{2}\right) + \beta$ with $|\beta| \leqq \delta\left(\frac{\varepsilon}{2}\right)$ is a translation number $\tau(\varepsilon)$, since according to Theorem II, the number β is a translation number $\tau\left(\frac{\varepsilon}{2}\right)$. So that in every interval of length $L\left(\frac{\varepsilon}{2}\right) + 2\delta\left(\frac{\varepsilon}{2}\right)$ lies an interval of length $2\delta\left(\frac{\varepsilon}{2}\right)$, and all the numbers in the latter interval are translation numbers $\tau(\varepsilon)$.

A more applicable form of the corollary is the somewhat weaker statement: To every ε there exists an L and a δ, so that for every arbitrarily chosen positive quantity $\eta \leqq \delta$ each interval $\alpha < x < \alpha + L$ of length L contains a translation number $\tau(\varepsilon)$ which is an integral multiple of η.

THE INVARIANCE OF ALMOST PERIODICITY RELATIVE TO SIMPLE OPERATIONS

48. We now prove a more difficult theorem:

THEOREM III. The sum $f(x) + g(x)$ of two almost periodic functions $f(x)$ and $g(x)$ is an almost periodic function.

Proof: It suffices to show that for each $\varepsilon > 0$ there exists a relatively dense set of common translation numbers $\tau(\varepsilon)$ of the two functions $f(x)$ and $g(x)$. In fact, every such number $\tau = \tau_f(\varepsilon) = \tau_g(\varepsilon)$ is obviously a translation number $\tau_{f+g}(2\varepsilon)$ because

$$|(f(x+\tau)+g(x+\tau)) - (f(x)+g(x))| \leqq |f(x+\tau)-f(x)| + |g(x+\tau)-g(x)| \leqq 2\varepsilon.$$

We use the corollary of §47 in its last form and determine the numbers L_f, δ_f and L_g, δ_g which correspond to the given number $\varepsilon/2$ according to this corollary. Let $L_0 = \mathrm{Max}(L_f, L_g)$ and $\eta = \mathrm{Min}(\delta_f, \delta_g)$. Then each interval $\alpha < x < \alpha + L_0$ of length L_0 contains a translation number $\tau_f\left(\frac{\varepsilon}{2}\right)$ as well as a translation number $\tau_g\left(\frac{\varepsilon}{2}\right)$, which are both multiples of η (see Fig. 6).

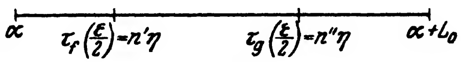

Fig. 6.

We consider all pairs of these translation numbers, i.e., all pairs $\left(\tau_f\left(\frac{\varepsilon}{2}\right), \tau_g\left(\frac{\varepsilon}{2}\right)\right)$ with $\tau_f = n'\eta, \tau_g = n''\eta$ and $|\tau_f - \tau_g| < L_0$. For each pair of this kind, we find

$$\tau_f - \tau_g = (n' - n'')\eta = n\eta,$$

where n is an integer. Since $|n\eta| < L_0$ there can be only finitely many distinct values of $n\eta$. Let these be $n_1\eta$, $n_2\eta, \ldots, n_Q\eta$ and let them be "represented" by the point pairs $(\tau_f^{(1)}, \tau_g^{(1)}), (\tau_f^{(2)}, \tau_g^{(2)}), \ldots, (\tau_f^{(Q)}, \tau_g^{(Q)})$ which are arbitrary, but, once chosen, are kept fixed. Further, let us write

$$\underset{q=1,2,\ldots,Q}{\mathrm{Max}} |\tau_f^{(q)}| = l$$

We shall show that each interval of length $L_0 + 2l$ contains at least one common translation number $\tau = \tau_f(\varepsilon) = \tau_g(\varepsilon)$.

Let $\alpha < x < \alpha + L_0 + 2l$ be an arbitrary interval of length $L_0 + 2l$. In the interval $\alpha + l < x < \alpha + L_0 + l$ of length L_0

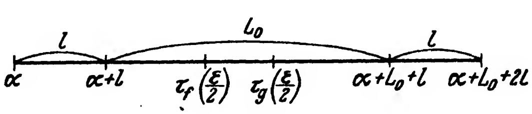

Fig. 7.

we choose (see Fig. 7) two translation numbers $\tau_f\left(\frac{\varepsilon}{2}\right) = n'\eta$ and $\tau_g\left(\frac{\varepsilon}{2}\right) = n''\eta$. Let $n_q\eta$ be that one of the above multiples of η, for which $\tau_f - \tau_g = n_q\eta$, and let $\tau_f^{(q)}, \tau_g^{(q)}$ be the previously chosen fixed "representation pair" ("Repraesentantenpaar") for this $n_q\eta$. Then $\tau_f - \tau_g = \tau_f^{(q)} - \tau_g^{(q)}$, or $\tau_f - \tau_f^{(q)} =$

$\tau_g - \tau_g^{(q)}$, and this last number

$$\tau = \tau_f - \tau_f^{(q)} = \tau_g - \tau_g^{(q)}$$

is what we have been looking for. In fact it is first of all a translation number of $f(x)$ as well as of $g(x)$, corresponding to ε, since it is the difference of two translation numbers corresponding to $\varepsilon/2$. Secondly, it lies in the interval $(\alpha, \alpha + L_0 + 2l)$, since τ_f belongs to the interval $(\alpha + l, \alpha + L_0 + l)$ and $|\tau_f^{(q)}| \leqq l$.

Corollary: The sum $f(x) = \sum_1^N p_n(x)$ of a finite number of continuous periodic functions $p_n(x)$ with arbitrary periods is almost periodic. In particular, every trigonometric polynomial,

$$s(x) = \sum_1^N a_n e^{i\lambda_n x}$$

is an almost periodic function.

(Since $-g(x)$ is almost periodic whenever $g(x)$ is, it follows from Theorem III that the difference $f(x) - g(x)$ is an almost periodic function).

THEOREM IV. The product $f(x)g(x)$ of two almost periodic functions $f(x)$ and $g(x)$ is also almost periodic.

Proof: It has been shown before, that the square of an almost periodic function is also almost periodic. The theorem in question, therefore, follows from Theorem III because of the identity

$$f(x)\,g(x) = \tfrac{1}{4}\{(f(x) + g(x))^2 - (f(x) - g(x))^2\}.$$

49. We have, finally, the following rather simple theorem.

THEOREM V. The limit function $f(x)$ of a sequence of almost periodic functions $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ uniformly convergent for $-\infty < x < \infty$ is also almost periodic.

Proof: We notice first that $f(x)$ is continuous. Now let $\varepsilon > 0$ be arbitrarily given, and let $N = N(\varepsilon)$ be chosen so large that

$$|f(x) - f_N(x)| \leqq \frac{\varepsilon}{3} \text{ for } -\infty < x < \infty.$$

Then every translation number $\tau = \tau_{f_N}\left(\frac{\varepsilon}{3}\right)$ is surely a translation number $\tau_f(\varepsilon)$. For we see that

$$|f(x+\tau)-f(x)|\leqq|f(x+\tau)-f_N(x+\tau)|+|f_N(x+\tau)-f_N(x)|+$$
$$|f_N(x)-f(x)|\leqq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

Since the translation numbers $\tau_{f_N}\left(\frac{\varepsilon}{3}\right)$ are relatively dense, the same is true of the translation numbers $\tau_f(\varepsilon)$.

<u>Corollary</u>. Every function $f(x)$, which can be approximated uniformly for all x by finite sums $s(x)=\sum_1^N a_n e^{i\lambda_n x}$ is almost periodic. In other words: <u>Every function $f(x)$ of the class $H\{s(x)\}$ is an almost periodic function.</u> Thus we have proved one part of our fundamental theorem. It is, however, the remaining part that offers the real difficulty.

THE MEAN VALUE THEOREM

50. Our next problem will be to lay the foundation for a theory of Fourier series of almost periodic functions. The key to this problem is Theorem IV together with the following theorem.

MEAN VALUE THEOREM. <u>For every almost periodic function there exists a mean value</u>

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x)\,dx=M\{f(x)\},$$

<u>i.e., the expression $\frac{1}{T}\int_0^T f(x)\,dx$ approaches a definite finite</u> limit as $T\to\infty$. We shall denote this by $M\{f(x)\}$.

<u>Remarks</u>: In case the function, $f(x)$, is purely periodic for example with period p, the theorem is trivial, and the mean value is in agreement with the earlier mean value

$$M\{f(x)\}=\frac{1}{p}\int_0^p f(x)\,dx\ .$$

In fact, we find that if we set $T=mp+r$ $(0\leqq r<p)$ then

$$\int_0^T f(x)\,dx=m\int_0^p f(x)\,dx+\int_{mp}^{mp+r} f(x)\,dx,$$

from which the desired relation

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x)\,dx=\frac{1}{p}\int_0^p f(x)\,dx$$

will follow, because the limit $\frac{m}{T}\to\frac{1}{p}$ as $T\to\infty$, (considering the fact that $f(x)$ is bounded).

Proof: We shall apply the general convergence principle. Accordingly, we let $\varepsilon>0$ be given arbitrarily. We have to show the existence of some number $T_0=T_0(\varepsilon)$ such that the inequality

$$\left|\frac{1}{T_1}\int_0^{T_1} f(x)dx-\frac{1}{T_2}\int_0^{T_2} f(x)dx\right|<2\varepsilon$$

holds for $T_1>T_0$, $T_2>T_0$. The proof of this will require several steps.

1. We denote by l_0 the length $L\left(\frac{\varepsilon}{2}\right)$ corresponding to a given $\varepsilon>0$. For a given fixed number $T>0$ and a given variable number α we wish to approximate the difference

$$\frac{1}{T}\int_0^T f(x)dx-\frac{1}{T}\int_\alpha^{\alpha+T} f(x)dx$$

This is done as follows on the basis of the almost-periodicity of $f(x)$:

In the interval $(\alpha,\alpha+l_0)$ we choose a translation number $\tau=\tau_f\left(\frac{\varepsilon}{2}\right)$ and obtain the difference under consideration in the form

$$\left\{\frac{1}{T}\int_0^T f(x)dx-\frac{1}{T}\int_\tau^{\tau+T} f(x)dx\right\}-\frac{1}{T}\int_\alpha^{\tau} f(x)dx+\frac{1}{T}\int_{\alpha+T}^{\tau+T} f(x)dx.$$

Now the first term $\{\ldots\}$ is numerically $\leq\frac{\varepsilon}{2}$, because $|f(x+\tau)-f(x)|\leq\frac{\varepsilon}{2}$, and each of the two remaining terms is numerically $\leq\frac{1}{T}l_0\Gamma$, where we denote by Γ the (finite) upper limit of $|f(x)|$ in $-\infty<x<\infty$. Thus we obtain:

$$\left|\frac{1}{T}\int_0^T f(x)dx-\frac{1}{T}\int_\alpha^{\alpha+T} f(x)dx\right|\leq\frac{\varepsilon}{2}+\frac{2l_0\Gamma}{T}. \tag{1}$$

2. Our next problem will be to estimate the difference

$$\frac{1}{T}\int_0^T f(x)\,dx - \frac{1}{nT}\int_0^{nT} f(x)\,dx$$

where $T > 0$ is given, and the positive integer n is arbitrary. This difference is simply the arithmetic mean of the n differences

$$\frac{1}{T}\int_0^T f(x)\,dx - \frac{1}{T}\int_{(\nu-1)T}^{\nu T} f(x)\,dx, \qquad \nu = 1, 2, \ldots, n;$$

each of which is according to (1), numerically $\leqq \frac{\varepsilon}{2} + \frac{2l_0\Gamma}{T}$. Thus the same inequality is true of the arithmetic mean of these differences. Thus we have

$$\left|\frac{1}{T}\int_0^T f(x)\,dx - \frac{1}{nT}\int_0^{nT} f(x)\,dx\right| \leqq \frac{\varepsilon}{2} + \frac{2l_0\Gamma}{T}. \tag{2}$$

3. Let us further prescribe $T_1 > 0$ and $T_2 > 0$ whose ratio T_1/T_2 we first take to be rational. Then we find two positive integers n_1 and n_2. such that $n_1T_1 = n_2T_2$. Now by (2) , however,

$$\left|\frac{1}{T_1}\int_0^{T_1} f(x)\,dx - \frac{1}{n_1T_1}\int_0^{n_1T_1} f(x)\,dx\right| \leqq \frac{\varepsilon}{2} + \frac{2l_0\Gamma}{T_1}$$

and

$$\left|\frac{1}{T_2}\int_0^{T_2} f(x)\,dx - \frac{1}{n_2T_2}\int_0^{n_2T_2} f(x)\,dx\right| \leqq \frac{\varepsilon}{2} + \frac{2l_0\Gamma}{T_2}.$$

Since $n_1T_1 = n_2T_2$, it follows that

$$\left|\frac{1}{T_1}\int_0^{T_1} f(x)\,dx - \frac{1}{T_2}\int_0^{T_2} f(x)\,dx\right| \leqq \varepsilon + 2l_0\Gamma\left(\frac{1}{T_1} + \frac{1}{T_2}\right). \tag{3}$$

Now, from considerations of continuity the relation (3) (proved at first only for rational values of the ratio T_1/T_2) must hold for arbitrary (positive) values of T_1 and T_2 .

After this preparation, we can now proceed immediately with the proof of the mean value theorem. Specifically, we choose $T_0 = \frac{4l_0\Gamma}{\varepsilon} = \frac{4\Gamma L\left(\frac{\varepsilon}{2}\right)}{\varepsilon}$, so that when $T_1 > T_0, T_2 > T_0$, we obviously have

$$\varepsilon + 2l_0\Gamma\left(\frac{1}{T_1} + \frac{1}{T_2}\right) < 2\varepsilon.$$

Thus from (3), it follows that

$$\left|\frac{1}{T_1}\int_0^{T_1} f(x)\,dx - \frac{1}{T_2}\int_0^{T_2} f(x)\,dx\right| < 2\varepsilon.$$

Remarks: For later applications it will be important to know exactly how large T should be chosen in order to make sure that the "finite" mean value $\frac{1}{T}\int_0^T f(x)\,dx$ actually represents the mean value $M\{f(x)\}$ to within a given degree of accuracy. Such a result can be obtained from the preceding proof. Now that the existence of the mean value $M\{f(x)\}$ is assured, we need only to perform the limiting process $n \to \infty$ in equation (2). We obtain

$$\left|\frac{1}{T}\int_0^T f(x)\,dx - M\{f(x)\}\right| \leqq \frac{\varepsilon}{2} + \frac{2l_0\Gamma}{T}$$

and from this follows the desired result

$$\left|\frac{1}{T}\int_0^T f(x)\,dx - M\{f(x)\}\right| < \varepsilon \text{ for } T > T_0 = \frac{4\Gamma L\left(\frac{\varepsilon}{2}\right)}{\varepsilon}. \tag{4}$$

In conclusion, it should now be mentioned that the mean values for almost periodic functions can be manipulated by the usual rules. For example $M\{c\,f(x)\} = c\,M\{f(x)\}$ (c an arbitrary complex constant) and

$$M\{(f_1(x) + f_2(x))\} = M\{f_1(x)\} + M\{f_2(x)\}.$$

It is important for applications to notice that if $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ denotes a sequence of almost periodic functions, which converges uniformly to the limit function $f(x)$ (which is eo ipso almost periodic) the limit equation is valid, or

$$M\{f(x)\} = \lim_{n\to\infty} M\{f_n(x)\}.$$

This follows immediately from the fact that for an arbitrary n, we have

$$M\{f(x)\} - M\{f_n(x)\} = M\{(f(x) - f_n(x))\}$$

and accordingly

$$|M\{f(x)\} - M\{f_n(x)\}| \leq M\{|f(x) - f_n(x)|\} \leq \underset{-\infty < x < \infty}{\text{upper bound}} |f(x) - f_n(x)|,$$

where the last quantity converges to zero as $n \to \infty$.

51. As the first application of inequality (4), we consider the set of all functions $f(x+a)$, where a denotes an arbitrary real constant. Each one of these functions is almost periodic and therefore possesses a mean value

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x+a)\,dx = M\{f(x+a)\}. \tag{5}$$

We wish to show that the limit holds uniformly with respect to a, or for a given $\varepsilon > 0$ a corresponding $T_0 = T_0(\varepsilon)$ can be chosen independently of a, so that the following inequality holds simultaneously for all values of a.

$$\left|\frac{1}{T}\int_0^T f(x+a)\,dx - M\{f(x+a)\}\right| < \varepsilon \text{ for } T > T_0.$$

This, however, is an immediate consequence of the above result, for indeed, the number $T_0 = \dfrac{4\Gamma L\left(\frac{\varepsilon}{2}\right)}{\varepsilon}$ is the same for all functions $f(x+a)$, since the upper limit of $|f(x+a)|$ equals Γ and $f(x+a)$ has precisely the same translation numbers as $f(x)$.

52. The result of § 51 can also be expressed in another form. To this end we first notice that for constant values of a, obviously

$$M\{f(x+a)\} = M\{f(x)\}$$

and indeed,

$$\frac{1}{T}\int_0^T f(x+a)\,dx = \frac{1}{T}\int_a^{a+T} f(x)\,dx = \frac{1}{T}\int_a^0 f(x)\,dx + \frac{1}{T}\int_0^T f(x)\,dx + \frac{1}{T}\int_T^{a+T} f(x)\,dx,$$

which approaches $M\{f(x)\}$ as $T \to \infty$, since the two other

terms converge to zero (each being bounded $\leq \frac{1}{T}|a|\Gamma$). The above equation, uniform with respect to a, can now be written as

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x+a)\,dx = M\{f(x)\}$$

or

$$\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} f(x)\,dx = M\{f(x)\}$$

We thus obtained the following theorem.

STRENGTHENED MEAN VALUE THEOREM: The following limit is approached uniformly with respect to a.

$$\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} f(x)\,dx = M\{f(x)\}.$$

This strengthened mean value theorem is a theorem that will be applied frequently below. It teaches us that the mean value of $f(x)$ over an arbitrary finite interval comes "close" to the actual mean value $M\{f(x)\}$, if the interval is chosen sufficiently large, independently of its position. As a trivial application, it might be mentioned that

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T}^{0} f(x)\,dx = M\{f(x)\} \quad \text{and} \quad \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x)\,dx = M\{f(x)\}.$$

53. In order not to have to break up our train of thought later, we prove here another application of the inequality (4).

Given $f(x)$, an arbitrary almost periodic function, we consider the set of all functions $F_x(t) = f(x+t)\overline{f(t)}$ with x an arbitrary real constant. According to Theorem IV each one of the functions $f(x+t)\overline{f(t)}$ is an almost periodic function of t and accordingly, each possesses a mean value

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x+t)\,\overline{f(t)}\,dt = \underset{t}{M}\{f(x+t)\,\overline{f(t)}\}.$$

We wish to show that this limit holds uniformly with respect to x, or that for a given $\varepsilon > 0$ a corresponding $T_0 = T_0(\varepsilon)$ of some sort can be chosen so that the inequality

$$\left|\frac{1}{T}\int_0^T f(x+t)\,\overline{f(t)}\,dt - \underset{t}{M}\{f(x+t)\,\overline{f(t)}\}\right| < \varepsilon \text{ for } T > T_0 \tag{6}$$

holds simultaneously for all x .

The proof is analogous to the preceding. According to the inequality (4) , the inequality

$$\left|\frac{1}{T}\int_0^T f(x+t)\,\overline{f(t)}\,dt - \underset{t}{M}\{f(x+t)\,\overline{f(t)}\}\right| < \varepsilon \text{ for } T > T_{0,x} = \frac{4\,\Gamma_x L_x\left(\frac{\varepsilon}{2}\right)}{\varepsilon},$$

holds when Γ_x and $L_x\left(\frac{\varepsilon}{2}\right)$ are the appropriate numbers corresponding to the function $F_x(t) = f(x+t)\overline{f(t)}$. Now since it is always true that

$$|f(x+t)\,\overline{f(t)}| \leqq \Gamma^2;$$

it follows that for all x , $\Gamma_x \leqq \Gamma^2$. Further for an arbitrary number τ

$$\begin{aligned} &F_x(t+\tau) - F_x(t) \\ &= f(x+t+\tau)\,\overline{f(t+\tau)} - f(x+t)\,\overline{f(t)} \\ &= [f(x+t+\tau) - f(x+t)]\cdot\overline{f(t+\tau)} + f(x+t)\cdot\left[\overline{f(t+\tau)} - \overline{f(t)}\right] \end{aligned}$$

and thus

$$\begin{aligned} &|F_x(t+\tau) - F_x(t)| \\ &\leqq |f(x+t+\tau) - f(x+t)|\cdot\Gamma + \Gamma\cdot\left|\overline{f(t+\tau)} - \overline{f(t)}\right| \\ &= \Gamma\{|f(x+t+\tau) - f(x+t)| + |f(t+\tau) - f(t)|\}, \end{aligned}$$

so that the translation number τ of $f(x)$ corresponding to $\varepsilon/4\Gamma$ is a translation number of $F_x(t)$ corresponding to $\varepsilon/2$. Thus the length $L\left(\frac{\varepsilon}{4\Gamma}\right)$ corresponding to $f(x)$ can be used as the length $L_x\left(\frac{\varepsilon}{2}\right)$. But now we have shown that the inequality (6) holds for

$$T > T_0 = \frac{4\Gamma^3 L\left(\frac{\varepsilon}{4\Gamma}\right)}{\varepsilon}$$

where $T_0 = T_0(\varepsilon)$ is independent of x .

54. In conclusion, for later application, we wish to investigate the mean value

$$g(x) = \underset{t}{M}\{f(x+t)\,\overline{f(t)}\}$$

as a function of x. We wish to show that $g(x)$ is an almost periodic function. To this end we first show that $g(x)$ is continuous. This follows immediately from the uniform continuity of $f(x)$. In fact, from

$$|f(x_1) - f(x_2)| \leqq \varepsilon \quad \text{for} \quad |x_1 - x_2| \leqq \delta = \delta(\varepsilon),$$

it follows that for $|h| \leqq \delta$

$$|g(x+h) - g(x)| = \left| \underset{t}{M} \left\{ [f(x+h+t) - f(x+t)] \cdot \overline{f(t)} \right\} \right| \leqq \varepsilon \Gamma.$$

That $g(x)$ is almost periodic follows from the fact that every translation number τ of $f(x)$ corresponding to ε/Γ is a translation number of $g(x)$ corresponding to ε, since

$$|g(x+\tau) - g(x)| = |\underset{t}{M}\{[f(x+\tau+t) - f(x+t)] \cdot \overline{f(t)}\}| \leqq \frac{\varepsilon}{\Gamma} \cdot \Gamma = \varepsilon$$

In just the same way a simpler result follows, namely, that the function

$$g_T(x) = \frac{1}{T}\int_0^T f(x+t)\,\overline{f(t)}\,dt$$

is an almost periodic function for each $T > 0$.

The mean value $M\{g(x)\}$ of $g(x)$ can be calculated in the following manner: Since the function $g_T(x)$ converges uniformly to $g(x)$ as $T \to \infty$, according to the result of § 53, then by an earlier remark,

$$M\{g(x)\} = \lim_{T\to\infty} M\{g_T(x)\}.$$

However, for each fixed T,

$$M\{g_T(x)\} = \lim_{X\to\infty} \frac{1}{X}\int_0^X g_T(x)\,dx = \lim_{X\to\infty} \frac{1}{X}\int_0^X \left(\frac{1}{T}\int_0^T f(x+t)\,\overline{f(t)}\,dt \right) dx,$$

and by change of the order of integration

$$= \lim_{X\to\infty} \frac{1}{T}\int_0^T \left(\frac{1}{X}\int_0^X f(x+t)\,\overline{f(t)}\,dx \right) dt = \lim_{X\to\infty} \frac{1}{T}\int_0^T \left(\frac{1}{X}\int_0^X f(x+t)\,dx \right) \overline{f(t)}\,dt.$$

In the last integral, however, the quantity $\frac{1}{X}\int_0^X f(x+t)\,dx$ approaches the mean value $\underset{x}{M}\{f(x+t)\} = \underset{x}{M}\{f(x)\}$ as $X \to \infty$ and, in fact, it does so uniformly in the interval $-\infty < t < \infty$, and all the more so in the finite interval of integration $0 \leqq t \leqq T$. It follows from this, however (since the pre-

sence of a bounded factor $\overline{f(t)}$ does not disturb the uniformity), that we can let $X \to \infty$ under the integral sign $\frac{1}{T}\int_0^T \ldots dt$ so that

$$M\{g_T(x)\} = \frac{1}{T}\int_0^T \underset{x}{M}\{f(x)\}\overline{f(t)}\,dt = \underset{x}{M}\{f(x)\}\frac{1}{T}\int_0^T \overline{f(t)}\,dt.$$

Finally from this it follows that

$$M\{g(x)\} = \lim_{T\to\infty} M\{g_T(x)\} = \underset{x}{M}\{f(x)\}\underset{t}{M}\{\overline{f(t)}\},$$

And thus (*) $$M\{g(x)\} = |M\{f(x)\}|^2.$$

This relation will be applied later.

Remarks: Formally we could have obtained immediately

$$M\{g(x)\} = \underset{x}{M}\{\underset{t}{M}\{f(x+t)\,\overline{f(t)}\}\} = \underset{t}{M}\{\underset{x}{M}\{f(x+t)\,\overline{f(t)}\}\}$$

$$= \underset{t}{M}\{\overline{f(t)}\,\underset{x}{M}\{f(x+t)\}\} = \underset{t}{M}\{\overline{f(t)}\,\underset{x}{M}\{f(x)\}\} = \underset{x}{M}\{f(x)\}\cdot\underset{t}{M}\{\overline{f(t)}\}.$$

We have interchanged the two mean value operators $\underset{x}{M}$ and $\underset{t}{M}$, which is in general not permissible since they require taking the mean over an infinite interval. The above proof shows us now, that in this particular case, it actually is true that

$$\underset{x}{M}\{\underset{t}{M}\{f(x+t)\,\overline{f(t)}\}\} = \underset{t}{M}\{\underset{x}{M}\{f(x+t)\,\overline{f(t)}\}\}.$$

We have derived this relation from the trivial interchange

$$\frac{1}{X}\int_0^X\left(\frac{1}{T}\int_0^T f(x+t)\,\overline{f(t)}\,dt\right)dx = \frac{1}{T}\int_0^T\left(\frac{1}{X}\int_0^X f(x+t)\,\overline{f(t)}\,dx\right)dt$$

by a process based on the above uniformity theorems.

THE CONCEPT OF THE FOURIER SERIES OF AN ALMOST PERIODIC FUNCTION. SETTING UP OF THE PARSEVAL'S EQUATION

55. We consider the system of <u>all pure vibrations</u> $e^{i\lambda x}$ <u>where λ is an arbitrary real number</u>. Every function $e^{i\lambda x}$, of this non-enumerable system is periodic but the periods are distinct. In contrast to the case of harmonic vibrations e^{inx} $(n = 0, \pm 1, \pm 2, \ldots)$ where we could restrict ourselves to

the finite interval $0 \leqq x \leqq 2\pi$ (since the functions e^{inx} have the common period 2π), we must consider the infinite interval $-\infty < x < \infty$. In this interval $-\infty < x < \infty$, our non-enumerable system $\{e^{i\lambda x}\}$ is a normalized orthogonal system in the sense that

$$M\{e^{i\lambda_1 x} e^{-i\lambda_2 x}\} = \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{i\lambda_1 x} e^{-i\lambda_2 x}\,dx = \begin{cases} 0 & \text{for} \quad \lambda_1 \neq \lambda_2 \\ 1 & \text{for} \quad \lambda_1 = \lambda_2 . \end{cases}$$

This is an immediate consequence of

$$\int_0^T e^{i\lambda_1 x} e^{-i\lambda_2 x}\,dx = \int_0^T e^{i(\lambda_1-\lambda_2)x}\,dx = \begin{cases} \dfrac{e^{i(\lambda_1-\lambda_2)T}-1}{i(\lambda_1-\lambda_2)} & \text{for} \quad \lambda_1 \neq \lambda_2 \\ T & \text{for} \quad \lambda_1 = \lambda_2 . \end{cases}$$

56. Let $f(x)$ now be an arbitrary almost periodic function; then for each real λ, the function $g(x)=f(x)e^{-i\lambda x}$ is also an almost periodic function, being the product of the almost periodic function $f(x)$ and the purely periodic function $e^{-i\lambda x}$. Consequently its mean value exists. Thus

$$M\{f(x)\,e^{-i\lambda x}\} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)\,e^{-i\lambda x}\,dx .$$

We denote this number $M\{f(x)e^{-i\lambda x}\}$ by $a(\lambda)$, so that we have defined a function $a(\lambda)$, corresponding to the almost periodic function $f(x)$ for $-\infty < \lambda < \infty$.

In our theory the following theorem (to be proved later) is of fundamental importance:

The function $a(\lambda) = M\{f(x)\,e^{-i\lambda x}\}$ is zero for all values of λ with the exception of an at most enumerable set of numbers λ.

It is this fact which will permit us to carry the theory of Fourier series over into the field of almost periodic functions.

57. As in § 14 we first derive the following formula.

Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be arbitrary, distinct, real numbers and let $c_1, c_2, \ldots, c_N$ be arbitrary complex numbers. Then

$$(1)\quad M\left\{\left|f(x) - \sum_1^N c_n\,e^{i\lambda_n x}\right|^2\right\} = M\{|f(x)|^2\} - \sum_1^N |a(\lambda_n)|^2 + \sum_1^N |c_n - a(\lambda_n)|^2 .$$

Proof: We notice, first, that the mean values encountered in this formula have meaning, for the two functions $f(x)$

and $f(x)-\sum_1^N c_n e^{i\lambda_n x}$ and likewise the squares of their absolute values) are almost periodic. The formula follows from the following calculation:

$$M\left\{\left|f(x)-\sum_1^N c_n e^{i\lambda_n x}\right|^2\right\}=M\left\{\left(f(x)-\sum_1^N c_n e^{i\lambda_n x}\right)\left(\overline{f(x)}-\sum_1^N \bar{c}_n e^{-i\lambda_n x}\right)\right\}$$

$$=M\{f(x)\overline{f(x)}\}-\sum_1^N \bar{c}_n M\{f(x)e^{-i\lambda_n x}\}-\sum_1^N c_n M\{\overline{f(x)}e^{i\lambda_n x}\}$$

$$+\sum_{n_1=1}^N\sum_{n_2=1}^N c_{n_1}\bar{c}_{n_2} M\{e^{i\lambda_{n_1}x}e^{-i\lambda_{n_2}x}\}$$

$$=M\{|f(x)|^2\}-\sum_1^N \bar{c}_n a(\lambda_n)-\sum_1^N c_n \overline{a(\lambda_n)}+\sum_{n=1}^N c_n\bar{c}_n$$

$$=M\{|f(x)|^2\}+\sum_1^N (c_n-a(\lambda_n))(\bar{c}_n-\overline{a(\lambda_n)})-\sum_1^N a(\lambda_n)\overline{a(\lambda_n)}$$

$$=M\{|f(x)|^2\}-\sum_1^N |a(\lambda_n)|^2+\sum_1^N |c_n-a(\lambda_n)|^2.$$

58. If in formula (1) the numbers $a(\lambda_n)$ are chosen for the constants c_n, then there follows the formula:

$$M\left\{\left|f(x)-\sum_1^N a(\lambda_n)e^{i\lambda_n x}\right|^2\right\}=M\{|f(x)|^2\}-\sum_1^N |a(\lambda_n)|^2. \tag{2}$$

Since the left hand member of formula (2) is obviously a real, non-negative number (being the mean value of a real, non-negative function), this formula yields the following inequality immediately:

$$\sum_1^N |a(\lambda_n)|^2 \leqq M\{|f(x)|^2\}. \tag{3}$$

Let us denote $M\{|f(x)|^2\}$ by C. Then for every positive d there can be only a finite number of values of λ for which $|a(\lambda)|>d$, (certainly fewer than C/d^2). We consider first the set of numbers λ, for which $|a(\lambda)|>1$, and denote these numbers for instance by

$$\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_{n_1};$$

Then we consider the set of those λ, for which $1\geqq|a(\lambda)|>\frac{1}{2}$, and we denote these by

$$\lambda_{n_1+1},\ \lambda_{n_1+2},\ \ldots,\ \lambda_{n_2};$$

After this, we consider the set of these numbers λ, for which $\frac{1}{2}\geqq|a(\lambda)|>\frac{1}{3}$ etc. In this way we obtain <u>the set of</u>

all λ for which $|a(\lambda)| > 0$, or for which $a(\lambda) \neq 0$, in the form of an ordered sequence, and thus we have shown that $a(\lambda)$ is zero for all λ with the exception of an at most enumerable set of values of λ. The exceptional values of λ (in any order) we shall denote by

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

and we shall call them the Fourier exponents of $f(x)$. The corresponding mean values

$$a(\Lambda_1) = A_1, \quad a(\Lambda_2) = A_2, \quad a(\Lambda_3) = A_3, \quad \ldots$$

we shall call the Fourier coefficients of $f(x)$. We shall further associate with $f(x)$, the finite or infinite series $\sum A_n e^{i\Lambda_n x}$, which we shall call its Fourier series and we shall indicate the formal correspondence by

$$f(x) \sim \sum A_n e^{i\Lambda_n x}.$$

Sometimes it is convenient to use the notation $f(x) \sim \sum a(\Lambda_n) e^{i\Lambda_n x}$, or even $f(x) \sim \sum a(\lambda) e^{i\lambda x}$, without expressly naming the exponents Λ_n. In the last notation, the series represents a formal sum over the non-enumerable set of all numbers $-\infty < \lambda < \infty$.

59. In the special case where $f(x)$ is purely periodic with period 2π, the new definition of Fourier series reduces to the previous one (except for such trivial considerations as the ordering of the terms or the permissibility of terms with coefficient zero). This is far from trivial; though, because of the periodicity of $f(x)e^{-inx}$, it is clear that if λ is an integer n

$$a(\lambda) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)\, e^{-inx}\, dx = \lim_{m\to\infty} \frac{1}{m\, 2\pi}\int_0^{m 2\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\, dx,$$

we must show more, namely, that when λ is not an integer

$$a(\lambda) = M\{f(x)\, e^{-i\lambda x}\} = 0 \ .$$

This follows most simply from the Weierstrass theorem, which says that for a given $\varepsilon > 0$ the function $f(x)$ can be written in the form

$$f(x) = \sum{}^{*} b_n e^{inx} + R(x)$$

where $|R(x)| \leqq \varepsilon$ for all x. From this, finally, it follows that

$$M\{f(x)\,e^{-i\lambda x}\} = \sum\nolimits^{*} b_n M\{e^{i(n-\lambda)x}\} + M\{R(x)\,e^{-i\lambda x}\} = M\{R(x)\,e^{-i\lambda x}\},$$

so that

$$|a(\lambda)| = |M\{R(x)\,e^{-i\lambda x}\}| \leqq \varepsilon\,.$$

Since this must hold for every $\varepsilon > 0$, $a(\lambda) = 0$ must vanish.

It should be noticed that the last proof can also be given by a rather primitive method without resorting to Weierstrass's theorem. If λ is not an integer, indeed, $e^{-i\lambda 2\pi} \neq 1$. Thus

$$\frac{1}{m2\pi}\int_0^{m2\pi} f(x)\,e^{-i\lambda x}\,dx = \frac{1}{m2\pi}\left\{\int_0^{2\pi} + \int_{2\pi}^{4\pi} + \cdots + \int_{(m-1)2\pi}^{m2\pi} f(x)\,e^{-i\lambda x}\,dx\right\}$$

$$= \frac{1}{m}\left\{1 + e^{-i\lambda 2\pi} + \cdots + e^{-i\lambda(m-1)2\pi}\right\}\frac{1}{2\pi}\int_0^{2\pi} f(x)\,e^{-i\lambda x}\,dx$$

$$= \frac{1}{m}\,\frac{1-e^{-i\lambda m2\pi}}{1-e^{-i\lambda 2\pi}}\,\frac{1}{2\pi}\int_0^{2\pi} f(x)\,e^{-i\lambda x}\,dx$$

and consequently,

$$a(\lambda) = \lim_{m\to\infty}\frac{1}{m\,2\pi}\int_0^{m2\pi} f(x)\,e^{-i\lambda x}\,dx = 0.$$

60. To elucidate further the concept of the Fourier series of an almost periodic function, we prove the following theorem.

SPECIAL THEOREM. Let the series $\sum a_n e^{i\lambda_n x}$ (with distinct real exponents $\lambda_1, \lambda_2, \lambda_3, \ldots$ and non zero coefficients) be uniformly convergent for $-\infty < x < \infty$ (or let it consist of only a finite number of terms), then this series is the Fourier series of its sum $f(x)$.

Proof: We know already that the sum $f(x)$ is an almost periodic function. We must prove that

$$a(\lambda) = M\{f(x)\,e^{-i\lambda x}\} = M\{\sum a_n e^{i\lambda_n x}\,e^{-i\lambda x}\} = \begin{cases} a_n & \text{for } \lambda = \lambda_n \\ 0 & \text{for } \lambda \neq \lambda_n \end{cases}$$

This follows, however, from the fact that the series $\sum a_n e^{i\lambda_n x} e^{-i\lambda x}$ converges uniformly for $-\infty < x < \infty$ (in case it has infinitely many terms). On that account, the mean value

process may be taken termwise.

Corollary: Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n, \ldots$ be a completely arbitrary (finite or infinite) sequence of real numbers. Then an almost periodic function can be constructed which has just these numbers Λ_n as Fourier exponents. (Thus, for instance, the sequence of exponents $\{\Lambda_n\}$ can have a finite accumulation point or even be everywhere dense in $-\infty < \lambda < \infty$.

For example, let the coefficient $a_n \neq 0$ corresponding to the given exponent Λ_n be so chosen that $\sum |a_n|$ converges, (e.g., $a_n = \frac{1}{2^n}$), then the series $\sum a_n e^{i\Lambda_n x}$ will certainly be a Fourier series, namely, the Fourier series of its sum $f(x)$.

61. Having made this interpolatory remark we resume our investigation. We consider again an arbitrary almost periodic function $f(x) \sim \sum A_n e^{i\Lambda_n x}$. We apply the inequality (3) of § 58 and we find by choosing $\lambda_n = \Lambda_n$, that for each N

$$\sum_1^N |A_n|^2 \leqq M\{|f(x)|^2\}.$$

Thus the series (with positive terms)

$$\sum |A_n|^2$$

is convergent (if it happens to contain infinitely many terms) and its sum is

$$\leqq M\{|f(x)|^2\}.$$

The fundamental fact will come out later that, exactly as in the case of purely periodic functions, the equality sign always holds in this relation, i.e., for an arbitrary almost periodic function, Parseval's equation holds:

$$\sum |A_n|^2 = M\{|f(x)|^2\}.$$

62. Exactly as in the theory of Fourier series of purely periodic functions, we can also give this theorem another formulation, one to which it owes its central position in the theory. Consider the formula

$$M\left\{|f(x) - \sum_1^N A_n e^{i\Lambda_n x}|^2\right\} = M\{|f(x)|^2\} - \sum_1^N |A_n|^2,$$

which we obtain from formula (2) of § 58 by choosing $\lambda_n = \Lambda_n$. One sees from it, that when the Fourier series has only a finite (say N_0) number of terms Parseval's equation reduces to

$$M\left\{|f(x) - \sum_1^{N_0} A_n e^{i\Lambda_n x}|^2\right\} = 0,$$

Moreover, in the general case, when the Fourier series contains infinitely many terms, Parseval's equation reduces to

$$\lim_{N\to\infty} M\left\{|f(x) - \sum_1^N A_n e^{i\Lambda_n x}|^2\right\} = 0.$$

This last formulation states that the partial sums of the Fourier series converge in the mean to $f(x)$.

63. Entirely independently of the rather deep Parseval equation, which settles the question of whether it is possible with arbitrary accuracy to approximate an almost periodic function in the mean by means of the partial sums of its Fourier series, there is one observation that we can make. We can conclude immediately from formula (1) of § 57 that if we wish to approximate an almost periodic function $f(x)$ in the mean by a finite trigonometric sum $\sum_1^N c_n e^{i\lambda_n x}$ then we should choose the terms $c_n e^{i\lambda_n x}$ from among the terms of the Fourier series of the function. For, first of all, the formula (1), shows that we should choose the exponents λ_n from among the Fourier exponents, since we get a better approximation (or a smaller mean-error) by omitting the terms $c_n e^{i\lambda_n x}$ for which λ_n is not a Fourier exponent. Secondly, formula (1) states that after the coefficients λ_n are chosen from among the Fourier exponents Λ_n, the coefficients c_n should be chosen as the corresponding Fourier coefficients A_n, for it is clear that any other choice of c_n would yield a poorer approximation.

CALCULATIONS WITH FOURIER SERIES

64. Theorems on the calculation with Fourier series of almost periodic functions are, for the most part, entirely analogous to the corresponding theorems for purely periodic functions.

From $f(x) \sim \sum A_n e^{i\Lambda_n x}$ it first follows that

(1) $$k f(x) \sim \sum k A_n \cdot e^{i\Lambda_n x}$$

(k an arbitrary complex constant). For

$$M\{k f(x) e^{-i\lambda x}\} = kM\{f(x) e^{-i\lambda x}\}.$$

(2) $$e^{i\Lambda x} f(x) \sim \sum A_n e^{i(\Lambda+\Lambda_n)x}.$$

For $$M\{e^{i\Lambda x} f(x) e^{-i\lambda x}\} = M\{f(x) e^{i(\Lambda-\lambda)x}\}.$$

(3) $$f(x+k) \sim \sum A_n e^{i\Lambda_n k} \cdot e^{i\Lambda_n x}$$

(k an arbitrary real constant). For

$$M\{f(x+k) e^{-i\lambda x}\} = e^{i\lambda k} M\{f(x+k) e^{-i\lambda(x+k)}\} = e^{i\lambda k} M\{f(x) e^{-i\lambda x}\}.$$

(4) $$\overline{f(x)} \sim \sum \overline{A_n}\, e^{-i\Lambda_n x}.$$

For

$$M\{\overline{f(x)}\, e^{-i\lambda x}\} = \overline{M\{f(x)\, e^{+i\lambda x}\}}.$$

From $f_1(x) \sim \sum A_n e^{i\Lambda_n x}$ and $f_2(x) \sim \sum B_n e^{iM_n x}$ it follows that

(5) $$f_1(x) + f_2(x) \sim \sum C_n e^{iN_n x},$$

where the last series is the result of the formal addition of the two series $\sum A_n e^{i\Lambda_n x}$ and $\sum B_n e^{iM_n x}$. (From formulas (1) and (5) we conclude that the Fourier series of the differences $f_1(x) - f_2(x)$ results from formal subtraction of the Fourier series of $f_1(x)$ and $f_2(x)$.

65. In contrast to the previously mentioned theorems, which are superficial, the following <u>general multiplication theorem</u> is quite deep.

The Fourier series of the product $f_1(x) f_2(x)$ is obtained by formal multiplication of the two series $\sum A_n e^{i\Lambda_n x}$ and $\sum B_n e^{iM_n x}$ Or, we have

(6) $$f_1(x) f_2(x) \sim \sum D_n e^{i\Pi_n x} \quad \text{with} \quad D_n = \sum_{\Lambda_p + M_q = \Pi_n} A_p B_q.$$

Or, to be exact, the exponent Π_n runs through the numbers of

the form $\Lambda_p + \mathrm{M}_q$, and the corresponding coefficient D_n is given by the indicated sum, which, if it is an infinite series, converges absolutely (omitting, of course, the terms $D_n e^{i\Pi_n x}$ for which $D_n = 0$).

We shall get to the proof of the theorem later (§§ 74-76).

66. It is important for later applications to find the Fourier series of the "folded" function

$$g(x) = M\{f(x+t)\overline{f(t)}\}$$

where $f(x) \backsim \sum A_n e^{i\Lambda_n x}$ is given. We know from § 54, that this function $g(x)$ is an almost periodic function. It will now be shown that

(7) $$g(x) \backsim \sum |A_n|^2 e^{i\Lambda_n x},$$

or that for each λ, the mean value is

$$M\{g(x)e^{-i\lambda x}\} = |a(\lambda)|^2 = |M\{f(x)e^{-i\lambda x}\}|^2$$

For $\lambda = 0$ this is the relation already proved in § 54.

(*) $$M\{g(x)\} = |M\{f(x)\}|^2.$$

From this the more general relation follows immediately by substituting $f(x)e^{-i\lambda x}$ for $f(x)$ so that $g(x)$ is replaced by the folded function

$$\underset{t}{M}\{f(x+t)e^{-i\lambda(x+t)}\overline{f(t)}e^{i\lambda t}\} = g(x)e^{-i\lambda x}$$

and the relation (*) becomes the more general relation

$$M\{g(x)e^{-i\lambda x}\} = |M\{f(x)e^{-i\lambda x}\}|^2 .$$

Remarks: Exactly as in the case of purely periodic functions the formula (7) is a special case of a more general formula, according to which, if $f_1(x) \backsim \sum A_n e^{i\Lambda_n x}$ and $f_2(x) \backsim \sum B_n e^{i\mathrm{M}_n x}$ are two arbitrary almost periodic functions, the function

$$g(x) = \underset{t}{M}\{f_1(x+t)f_2(t)\}$$

is also almost periodic and has as its Fourier series the series

(8) $$g(x) \backsim \sum E_n e^{i\mathrm{P}_n x}$$

where P_n takes on those values Λ_n for which $-\Lambda_n$ is one of the numbers M_n; where $E_n = A_p B_q$, and where $\mathrm{P}_n = \Lambda_p = -\mathrm{M}_q$.

67. Let a series of almost periodic functions

$$f_m(x) \sim \sum A_n^{(m)} e^{i\Lambda_n^{(m)} x}$$

be given, which approaches a limit function $f(x)$ uniformly for all x as $m \to \infty$. Then

$$f(x) = \lim_{m \to \infty} f_m(x) \sim \lim_{m \to \infty} \sum A_n^{(m)} e^{i\Lambda_n^{(m)} x} \tag{9}$$

This means that for a fixed λ, this limit equation holds:

$$M\{f(x)\, e^{-i\lambda x}\} = \lim_{m \to \infty} M\{f_m(x)\, e^{-i\lambda x}\}$$

(and in fact, holds uniformly for all λ)

The proof follows immediately from the fact that

$$M\{f(x)\, e^{-i\lambda x}\} - M\{f_m(x)\, e^{-i\lambda x}\} = M\{[f(x) - f_m(x)]\, e^{-i\lambda x}\} .$$

If an $\varepsilon > 0$ is given arbitrarily, then there exists a $M = M(\varepsilon)$ such that for $m > M$,

$$|f(x) - f_m(x)| \leqq \varepsilon \text{ for all } x.$$

From this it follows, however, that

$$|M\{f(x)e^{-i\lambda x}\} - M\{f_m(x)e^{-i\lambda x}\}| \leqq \varepsilon \quad \text{for } m > M \text{ and all } \lambda.$$

68. In conclusion we take up the question of the indefinite integral $F(x)$ of an almost periodic function $f(x) \sim \sum A_n e^{i\Lambda_n x}$. Since the various indefinite integrals differ only by an additive constant, it is sufficient to consider only one of them, say

$$F(x) = \int_0^x f(\xi)\, d\xi .$$

We wish to investigate whether this function $F(x)$ is almost periodic, and if it is, to determine its Fourier series.

Here the analogy to purely periodic functions immediately breaks down. We should expect $F(x)$ to be almost periodic if and only if the Fourier series of $f(x)$ contains no constant term; that is, if $M\{f(x)\} = 0$. Moreover, in this case we should expect the formula

$$F(x) \sim C + \sum \frac{A_n}{i\Lambda_n} e^{i\Lambda_n x} \qquad C = \text{const.} \tag{10}$$

to hold. It can be shown, however, that the condition $M\{f(x)\} = 0$ although necessary, is <u>no longer</u> sufficient

for the almost-periodicity of $F(x)$. If, however, $F(x)$ is almost periodic, (and an extremely simple necessary and sufficient condition for this will be given in § 69), then the formula (10) is valid.

Proof: One sees immediately, that for the almost-periodicity of $F(x)$ it is necessary that

$$M\{f(x)\}=0.$$

For if $M\{f(x)\}=c\neq 0$, i.e.,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x)\,dx=\lim_{T\to\infty}\frac{F(T)}{T}=c,$$

then $F(x)$ is not bounded. And furthermore, if the integral $F(x)$ is almost periodic, its Fourier series is given by the formula (10), as seen by integration by parts. In fact, for each $\lambda\neq 0$

$$M\{F(x)\,e^{-i\lambda x}\}=\lim_{T\to\infty}\frac{1}{T}\int_0^T F(x)\,e^{-i\lambda x}\,dx$$

$$=\lim_{T\to\infty}\frac{1}{T}\left(\left[F(x)\frac{e^{-i\lambda x}}{-i\lambda}\right]_0^T+\frac{1}{i\lambda}\int_0^T f(x)\,e^{-i\lambda x}\,dx\right)$$

$$=-\frac{1}{i\lambda}\lim_{T\to\infty}\frac{F(T)}{T}e^{-i\lambda T}+\frac{1}{i\lambda}M\{f(x)\,e^{-i\lambda x}\}=\frac{1}{i\lambda}M\{f(x)\,e^{-i\lambda x}\},$$

since the presupposed almost-periodity of $F(x)$ implies the equation $M\{f(x)\}=0$, i.e., that $\lim\limits_{T\to\infty}\frac{F(T)}{T}=0$.

We must still show that the condition $M\{f(x)\}=0$ is not sufficient for the almost-periodicity of $F(x)$. This follows, immediately, however, from what was just proven. For, if we choose a sequence of coefficients $A_n\neq 0$ and a sequence of distinct exponents $\Lambda_n\neq 0$, so that $\sum\limits_1^\infty|A_n|$ converges and at the same time the series $\sum\limits_1^\infty\left|\frac{A_n}{\Lambda_n}\right|^2$ diverges (e.g., $A_n=\Lambda_n=\frac{1}{2^n}$) then the uniformly convergent series $\sum A_n e^{i\Lambda_n x}$ obviously satisfies the condition $M\{f(x)\}=0$ since $\Lambda_n\neq 0$ for all n, and, indeed, its Fourier series is the series $\sum A_n e^{i\Lambda_n x}$. Its integral $F(x)$ is, however, by no means

almost periodic, for then, we should have

$$F(x) \sim C + \sum \frac{A_n}{i\Lambda_n} e^{i\Lambda_n x};$$

but the series on the right hand side is not the Fourier series of an almost periodic function, since the series

$\sum_{1}^{\infty} \left|\frac{A_n}{\Lambda_n}\right|^2$ diverges.

69. We prove a very interesting theorem, which provides a necessary and sufficient condition for the almost periodicity of the integral $F(x)$. This condition states that the mere boundedness of $F(x)$ suffices for the almost-periodicity of this function. That is, the following theorem is valid.

THEOREM. For the almost-periodicity of the integral $F(x)$ it is necessary and sufficient that $F(x)$ be bounded.

Proof: Obviously $f(x)$ and also $F(x)$ may be taken as real. By assumption the function $F(x)$ is bounded. We denote its lower and upper bounds by k_1 and k_2; (where we can take $k_1 < k_2$ since if $k_1 = k_2$ the function $F(x)$ is constant).

Let $\varepsilon > 0$ be given arbitrarily. The proof will result from the fact that a number $\varepsilon_1 = \varepsilon_1(\varepsilon, f(x))$ can be so determined that every translation number of the given function $f(x)$ corresponding to ε_1 is also a translation number of $F(x)$ corresponding to ε.

To this end we choose two fixed values x_1 and x_2 such that

$$F(x_1) < k_1 + \frac{\varepsilon}{6} \quad \text{and} \quad F(x_2) > k_2 - \frac{\varepsilon}{6}$$

We shall henceforth denote the smaller of them by ξ and the distance $|x_2 - x_1|$ by d. We determine the length $l_0 = L\left(\frac{\varepsilon}{6d}\right)$ so that in every interval of this length at least one translation number τ of $f(x)$ exists, corresponding to $\varepsilon/6d$. First we shall show that the oscillations of $F(x)$ display a certain "regularity", namely, that in each interval $(\alpha, \alpha + L_0)$ of length $L_0 = l_0 + d$ two values z_1 and z_2 can be found such that

$$F(z_1) < k_1 + \frac{\varepsilon}{2}, \quad \text{and} \quad F(z_2) > k_2 - \frac{\varepsilon}{2} \tag{1}$$

In fact, according to the definition of the length l_0 we can choose a translation number $\tau = \tau\left(\frac{\varepsilon}{6d}\right)$ of the function $f(x)$ so that the number $\xi + \tau$ falls in the interval $(\alpha, \alpha + l_0)$ and such that the two numbers $x_1 + \tau = z_1$ and $x_2 + \tau = z_2$ lie in the larger interval $(\alpha, \alpha + L_0)$. Then this relation is valid:

$$F(z_2)-F(z_1)=F(x_2)-F(x_1)+\int_{z_1}^{z_2} f(y)\,dy-\int_{x_1}^{x_2} f(y)\,dy=F(x_2)-F(x_1)+\int_{x_1}^{x_2}(f(y+\tau)-f(y))\,dy.$$

Accordingly

$$F(z_2) - F(z_1) \geqq F(x_2) - F(x_1) - d\frac{\varepsilon}{6d} > k_2 - k_1 - \frac{2\varepsilon}{6} - \frac{\varepsilon}{6} = k_2 - k_1 - \frac{\varepsilon}{2}.$$

But, by the definition of k_1 and k_2, the inequality

$$F(z_2) - F(z_1) > k_2 - k_1 - \frac{\varepsilon}{2}$$

is possible only when $F(z_1)$ and $F(z_2)$ satisfy the desired inequalities (1).

We now assert that the "small" number $\varepsilon_1 = \frac{\varepsilon}{2L_0}$ has the property mentioned above, and therefore that each "fine" translation number τ (i.e., one belonging to ε_1) is a translation number of the function $F(x)$ belonging to ε. This means that

$$(2) \qquad |F(x+\tau) - F(x)| \leqq \varepsilon$$

for all x. To prove this, we use a simple artifice, and show that the two inequalities

$$(3a) \qquad F(x+\tau) - F(x) \geqq -\varepsilon$$

and

$$(3b) \qquad F(x+\tau) - F(x) \leqq \varepsilon$$

hold.

(a) For the proof of the inequality (3a) we choose a number z_1 in the interval $(x, x + L_0)$ where x is arbitrarily given and z_1 satisfies the condition $F(z_1) < k_1 + \frac{\varepsilon}{2}$. Then we have

$$\begin{aligned} F(x+\tau) - F(x) &= F(z_1+\tau) - F(z_1) + \int_x^{x+\tau} f(y)\,dy - \int_{z_1}^{z_1+\tau} f(y)\,dy \\ &= F(z_1+\tau) - F(z_1) + \int_x^{z_1} f(y)\,dy - \int_{x+\tau}^{z_1+\tau} f(y)\,dy \end{aligned}$$

$$> k_1 - \left(k_1 + \frac{\varepsilon}{2}\right) - \left| \int_x^{z_1} (f(y+\tau) - f(y))\, dy \right| > -\frac{\varepsilon}{2} - L_0 \frac{\varepsilon}{2L_0} = -\varepsilon.$$

(b) The proof of the inequality (3b) follows similarly, except that this time we use a point z_2 in the interval $(x, x+L_0)$ for which $F(z_2) > k_2 - \frac{\varepsilon}{2}$
We then obtain analogously

$$F(x+\tau) - F(x) = F(z_2+\tau) - F(z_2) + \int_x^{z_2} f(y)\, dy - \int_{x+\tau}^{z_2+\tau} f(y)\, dy$$

$$< k_2 - \left(k_2 - \frac{\varepsilon}{2}\right) + \left| \int_x^{z_2} (f(y+\tau) - f(y))\, dy \right| < \frac{\varepsilon}{2} + L_0 \frac{\varepsilon}{2L_0} = \varepsilon.$$

This concludes the proof.

THE UNIQUENESS THEOREM
ITS EQUIVALENCE WITH PARSEVAL'S EQUATION

70. As we shall later show for almost periodic functions, an almost periodic function is always uniquely determined by its Fourier series, or, that two distinct functions $f_1(x)$ and $f_2(x)$ always determine two distinct Fourier series. Exactly as in the case of purely periodic functions this uniqueness theorem can be expressed in another form. To this end we consider the function $f(x) = f_1(x) - f_2(x)$. As we noticed earlier, the Fourier series of this function $f(x)$ results from formal subtraction of the two Fourier series corresponding respectively to $f_1(x)$ and $f_2(x)$. Thus, the two functions $f_1(x)$ and $f_2(x)$ have the same Fourier series if, and only if, the Fourier series corresponding to $f(x)$ has no non-zero terms, or

$$a(\lambda) = M\{f(x)\, e^{-i\lambda x}\} = 0$$

for all λ. Thus the uniqueness theorem can be formulated as follows: There is no almost periodic function $f(x)$, which does not identically vanish, for which $a(\lambda) = 0$ for all λ.

Using a terminology similar to that of harmonic oscillation e^{inx} with respect to the non-enumerable totality of all oscillations $e^{i\lambda x}$, we can express the uniqueness theorem in another form. Thus, we say that there exists no almost periodic function $f(x)$, not identically zero, such

that the orthogonality relation

$$M\{f(x)e^{-i\lambda x}\} = 0$$

is satisfied for each λ. Thus we also say that <u>the system $\{e^{i\lambda x}\}$ orthogonal in the interval $-\infty < x < \infty$ is complete in the class of almost periodic functions</u>. (We could have also restricted our considerations to normalized functions at this point for we shall show in §72 that if an arbitrary almost periodic function does not vanish identically, we can multiply it by a suitably chosen constant and make the mean value of its square unity.)

It should be expressly mentioned, however, that it would <u>not</u> have been permissible to omit the words "in the class of almost periodic functions." In fact, there exist some rather simple functions $f(x)$, that are not <u>almost periodic</u> such that, for each λ

$$\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} f(x)\, e^{-i\lambda x}\, dx = 0$$

is satisfied, (even uniformly with respect to a), without the function vanishing identically. This happens, not only in such trivial cases as, those for example, in which $f(x) \to 0$ as $|x| \to \infty$, but also, as can be seen by simple approximations, for functions like e^{ix^2} and $e^{i\sqrt{x}}$. The first of these functions has very "rapid" oscillations and the second has very "slow" oscillations as $|x| \to \infty$ while the absolute value of each is even constant (in fact unity).

<u>Example</u>: If a purely periodic function $f(x)$ of period 2π is thought of as almost periodic, then we know (according to §59) that its Fourier exponents Λ_n are all <u>integral</u>. From the uniqueness theorem it would follow conversely that an almost periodic function $f(x) \sim \sum A_n e^{i\Lambda_n x}$ with integral exponents is necessarily a purely periodic function of period 2π. In fact the Fourier series of the function $f(x+2\pi)$, is determined from the Fourier series of $f(x)$ by applying formula (3) of § 64, or by the mere substitution of $x + 2\pi$ for x in that Fourier series. Thus we find that the new series is identical with the original series $\sum A_n e^{i\Lambda_n x}$.

71. The uniqueness theorem is a really deep theorem,

whose proof we shall defer until later. We shall, in the meantime, content ourselves with showing that it is fully equivalent to the other fundamental theorem of the theory, namely Parseval's equation, which states that for an arbitrary almost periodic function $f(x) \sim \sum A_n e^{i\Lambda_n x}$ the relation

$$\sum |A_n|^2 = M\{|f(x)|^2\}$$

holds. The proof of this equivalence is completely analogous to the corresponding proof for periodic functions.

1. The uniqueness theorem follows from Parseval's equation.

Proof: Let $f(x)$ be an almost periodic function, whose Fourier series has no terms whatsoever. Then from Parseval's equation $M\{|f(x)|^2\} = \sum |A_n|^2$, it follows that $M\{|f(x)|^2\} = 0$. Our problem is to show on the basis of $M\{|f(x)|^2\} = 0$ that $f(x)$ vanishes identically. For purely periodic functions the corresponding conclusion is trivial, since, in fact, the mean value of a non negative continuous function over a finite interval can vanish only if the function vanishes identically. For almost periodic functions, where we are taking a mean value over an infinite interval the conclusion is (fortunately) still true, although not immediate. We shall defer the proof to the next paragraph in order to prove for later use some further theorems in the same connection.

2. Parseval's theorem follows from the uniqueness theorem.

Proof: Let $f(x) \sim \sum A_n e^{i\Lambda_n x}$ be an arbitrary almost periodic function. We form the following function (which, by formula (7) in § 66 is itself almost periodic):

$$g(x) = M\{f(x+t)\overline{f(t)}\} \sim \sum |A_n|^2 e^{i\Lambda_n x}.$$

Since the series $\sum |A_n|^2$ converges (with a sum $\leqq M\{|f(x)|^2\}$, by § 61), then the last term $\sum |A_n|^2 e^{i\Lambda_n x}$ is uniformly convergent for all x (when it contains, as it does in general, infinitely many terms). Thus (by the theorem of § 60) the latter series is the Fourier series of its (eo ipso) almost periodic sum $s(x)$. The two almost periodic functions $g(x)$ and $s(x)$ accordingly have the same Fourier series (namely $\sum |A_n|^2 e^{i\Lambda_n x}$) and therefore, by the uniqueness

theorem, $g(x) = s(x)$, i.e.,

$$g(x) = \sum |A_n|^2 e^{i\Lambda_n x}.$$

If, in this equation, we choose in particular $x = 0$, we get

$$g(0) = \underset{t}{M}\{f(t)\overline{f(t)}\} = M\{|f(t)|^2\} = \sum |A_n|^2,$$

or Parseval's equation.

72. From the preceeding section we still need to prove the following theorem:

THEOREM. <u>An almost periodic function</u> $f(x)$, <u>for which</u> $M\{|f(x)|^2\} = 0$, <u>must vanish identically, i.e., for all</u> x

<u>Proof</u>: We give an indirect proof, i.e., we assume that $f(x)$ is not identically zero, and then we show that $M\{|f(x)|^2\} > 0$. By our assumption, a positive number α must exist such that in some point, say x_0, the inequality $|f(x_0)| \geqq \alpha$ holds. Now, by a previous remark (see § 47) there exists a length corresponding to the number $\varepsilon = \frac{\alpha}{2}$. Call this length $L(\varepsilon) = L\left(\frac{\alpha}{2}\right)$. There exists a positive number $\delta(\varepsilon) = \delta\left(\frac{\alpha}{2}\right)$, such that each interval of length L contains a complete interval of length δ whose points τ are <u>all</u> translation numbers $\tau_f(\varepsilon) = \tau_f\left(\frac{\alpha}{2}\right)$ and thus satisfy the equation:

$$|f(x_0+\tau)| \geqq |f(x_0)| - |f(x_0+\tau) - f(x_0)| \geqq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}.$$

Thus the integral of $|f(x)|^2$ extended over an arbitrary interval of length $L\left(\frac{\alpha}{2}\right)$ is surely $\geqq \delta\left(\frac{\alpha}{2}\right)\cdot\left(\frac{\alpha}{2}\right)^2$, and the mean value $M\{|f(x)|^2\}$ satisfies the inequality

$$M\{|f(x)|^2\} = \lim_{m\to\infty} \frac{1}{mL\left(\frac{\alpha}{2}\right)} \int_0^{mL\left(\frac{\alpha}{2}\right)} |f(x)|^2\,dx \geqq \frac{\delta\left(\frac{\alpha}{2}\right)\cdot\left(\frac{\alpha}{2}\right)^2}{L\left(\frac{\alpha}{2}\right)},$$

and therefore it is positive, contrary to assumption.

<u>Remarks</u>: We say that a set $\{\varphi(x)\}$ of almost periodic functions is <u>majorized</u> by the <u>almost periodic function</u> $f(x)$, or that it is <u>majorizable with majorant</u> $f(x)$, <u>if each translation number</u> $\tau = \tau_f(\varepsilon)$ <u>of</u> $f(x)$ <u>corresponding to an arbi-</u>

trary number $\varepsilon > 0$ is at the same time a translation number $\tau_\varphi(\varepsilon)$ corresponding to ε for each function $\varphi(x)$ of our set. Then the above proof shows the truth of the following assertion:

Let $\{\varphi(x)\}$ be a set of almost periodic functions which is majorized by the almost periodic function $f(x)$. Then there exists for each $\alpha > 0$ a number $\beta = \beta(\alpha) > 0$, such that the inequality

$$M\{|\varphi(x)|^2\} \geqq \beta$$

holds for every function $\varphi(x)$ of our set for which

$$\underset{-\infty<x<\infty}{\text{upper limit}} |\varphi(x)| > \alpha .$$

In fact, we can take for $\beta = \beta(\alpha)$ the value just found,

$$\beta = \frac{\delta\left(\frac{\alpha}{2}\right)\cdot\left(\frac{\alpha}{2}\right)^2}{L\left(\frac{\alpha}{2}\right)},$$

where $L\left(\frac{\alpha}{2}\right)$ and $\delta\left(\frac{\alpha}{2}\right)$ correspond to $f(x)$, the majorant of the set; and accordingly could be used all the more for the individual functions $\varphi(x)$ of the set.

73. An immediate corollary of the last remark is the following theorem: Let $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$ be a majorizable sequence of almost periodic functions (majorized by an almost periodic function $f(x)$), and let it only be assumed that

$$M\{|\varphi_n(x)|^2\} \to 0 \quad \text{for} \quad n \to \infty;$$

Then $\varphi_n(x)$ converges to 0 as $n \to \infty$, and indeed uniformly in the interval $-\infty < x < \infty$, i.e.,

$$\underset{-\infty<x<\infty}{\text{up}} |\varphi_n(x)| \to 0 \quad \text{for} \quad n \to \infty .$$

In fact, for an arbitrarily given $\alpha > 0$ we can find a number $\beta = \beta(\alpha) > 0$ such that

$$\underset{-\infty<x<\infty}{\text{upper limit}} |\varphi_n(x)| \leqq \alpha, \quad \text{as long as} \quad M\{|\varphi_n(x)|^2\} < \beta$$

and the last inequality $M\{|\varphi_n(x)|^2\} < \beta$ is surely satisfied for all sufficiently large values of n.

We shall later apply these remarks in the following form: Let $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ be a sequence of almost periodic functions which are majorized by the almost

periodic function $f(x)$. Then it follows that $f_n(x)$ converges uniformly as $n \to \infty$ in case it is known that $f_n(x)$ approaches $f(x)$ in the mean as $n \to \infty$. i.e., whenever

$$M\{|f(x) - f_n(x)|^2\} \to 0 \quad \text{for} \quad n \to \infty.$$

As proof we have only to set $f(x) - f_n(x) = \varphi_n(x)$. Then the sequence $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$ is obviously majorizable, namely with majorant $2f(x)$. (For $\tau_{2f}(\varepsilon) = \tau_f(\varepsilon/2) = \tau_{f_n}(\varepsilon/2) = \tau_{f-f_n}(\varepsilon)$), and thus from the hypothesis $M\{|\varphi_n(x)|^2\} \to 0$ as $n \to \infty$, it follows that $\varphi_n(x)$ converges uniformly to zero as $n \to \infty$.

THE MULTIPLICATION THEOREM

74. Before we prove the uniqueness theorem and with it Parseval's equation, we wish to give a proof of the multiplication theorem (6) in § 65 on the basis of these last two theorems. Specifically, we wish to show that the multiplication theorem is completely equivalent with Parseval's equation.

The general multiplication theorem states: If $f_1(x) \sim \sum A_n e^{i\Lambda_n x}$ and $f_2(x) \sim \sum B_n e^{iM_n x}$ are two arbitrary almost periodic functions then

(6) $$f_1(x) f_2(x) \sim \sum D_n e^{i\Pi_n x}, \quad D_n = \sum_{\Lambda_p + M_q = \Pi_n} A_p B_q.$$

As already mentioned, this means, that the exponent Π_n runs through all numbers of the form $\Lambda_p + M_q$ and, further, that the corresponding coefficient D_n is given by the indicated sum where the series, if it has an infinite number of terms, is absolutely convergent. (Naturally we omit terms $D_n e^{i\Pi_n x}$ with $D_n = 0$.)

It is a matter of showing that for every real ν

$$M\{f_1(x) f_2(x) e^{-i\nu x}\} = \sum_{\Lambda_p + M_q = \nu} A_p B_q$$

(where the right side is taken to be zero for an empty sum), and of showing also, that the right hand series converges absolutely when it has infinitely many terms. We note at this point that the last part of the theorem is trivial because of the elementary inequality $|ab| \leq \frac{1}{2}(|a|^2 + |b|^2)$. By this inequality the series $\sum_{\Lambda_p + M_q = \nu} |A_p B_q|$ is obviously majorized by the

series $\frac{1}{2}\sum_{\Lambda_p+M_q=\nu}(|A_p|^2+|B_q|^2)$. The convergence of this latter series, in turn, follows immediately from the convergence of the two series $\sum|A_n|^2$ and $\sum|B_n|^2$, since (because of the condition $\Lambda_p+M_q=\nu$) the same coefficient A_n or B_n never enters in two different products of the form A_pB_q.

We notice further that it suffices to consider the special case $\nu=0$ where we work with the constant term of the Fourier series or with the equation

$$(*)\qquad M\{f_1(x)\,f_2(x)\}=\sum_{\Lambda_p+M_q=0}A_pB_q.$$

If this equation is indeed valid, we can simply replace $f_2(x)$ by the function $f_2(x)e^{-i\nu x}\sim\sum B_n e^{i(M_n-\nu)x}$ and obtain the more general equation

$$M\{f_1(x)\,f_2(x)\,e^{-i\nu x}\}=\sum_{\Lambda_p+(M_q-\nu)=0}A_pB_q=\sum_{\Lambda_p+M_q=\nu}A_pB_q.$$

75. We now go on to the proof itself and show first

1. Parseval's equation follows from the multiplication theorem.

If one makes the special choice $f_2(x)=\overline{f_1(x)}\sim\sum\overline{A}_n e^{-i\Lambda_n x}$ then from (*) it follows that

$$M\{f_1(x)\,\overline{f_1(x)}\}=\sum_{\Lambda_p+(-\Lambda_q)=0}A_p\overline{A}_q,$$

which is nothing but Parseval's equation

$$M\{|f_1(x)|^2\}=\sum|A_n|^2.$$

2. The Multiplication theorem follows from Parseval's equation.

The proof depends on the same simple trick as the one we used in the theory of purely periodic functions, namely, the use of the elementary identity.

$$uv=\tfrac{1}{4}\{|u+\bar v|^2-|u-\bar v|^2+i|u+i\bar v|^2-i|u-i\bar v|^2\}.$$

If we apply this to the product $f_1(x)\,f_2(x)$, it yields

$$M\{f_1f_2\}=\tfrac{1}{4}[M\{|f_1+\bar f_2|^2\}-M\{|f_1-\bar f_2|^2\}+iM\{|f_1+i\bar f_2|^2\}-iM\{|f_1-i\bar f_2|^2\}]$$

If we now apply Parseval's equation to the functions $f_1(x)+\overline{f_2(x)}$, $f_1(x)-\overline{f_2(x)}$, $f_1(x)+i\overline{f_2(x)}$ and $f_1(x)-i\overline{f_2(x)}$, and using their Fourier series which can be obtained from formulas (1), (4) and (5) of § 64, and then momentarily setting

$$a(\lambda)=M\{f_1(x)\,e^{-i\lambda x}\}\quad\text{and}\quad b(\lambda)=M\{f_2(x)\,e^{-i\lambda x}\}$$

we find that the right hand member of the above relation becomes

$$= \tfrac{1}{4}[\sum|a(\lambda)+\overline{b(-\lambda)}|^2 - \sum|a(\lambda)-\overline{b(-\lambda)}|^2 + i\sum|a(\lambda)+i\overline{b(-\lambda)}|^2 - i\sum|a(\lambda)-i\overline{b(-\lambda)}|^2]$$

where each summation is over those values of λ for which the summand is positive.

Here we can apply the same identity, only in the reverse direction and obtain

$$M\{f_1 f_2\} = \sum a(\lambda)\, b(-\lambda),$$

where the summation is interpreted as above. This, however, is just the desired relation (*).

76. Finally we notice that we can still prove the multiplication theorem exactly as for purely periodic functions, that is, without using the above artifice but rather by starting with the uniqueness theorem. We have only to draw upon formula (8) of § 66 according to which the (almost periodic) function

$$g(x) = \underset{t}{M}\{f_1(x+t)\, f_2(t)\}$$

has the Fourier series

$$g(x) \sim \sum E_n e^{i\mathrm{P}_n x}$$

where P_n takes on the values $\varLambda_n$ for which $-\varLambda_n$ occurs among the set of numbers M_n and where $E_n = A_p B_q$, if $\mathrm{P}_n = \varLambda_p = -\mathrm{M}_q$.

The series converges uniformly, the series $\sum E_n$ being the absolutely convergent series $\sum\limits_{\varLambda_p+\mathrm{M}_q=0} A_p B_q$. Thus the sum of the Fourier series must be an almost periodic function $s(x)$, whose Fourier series is exactly the series $\sum E_n e^{i\mathrm{P}_n x}$, and which must coincide, by the uniqueness theorem, with the function $g(x)$. Now, in the relation just derived

$$g(x) = \sum E_n e^{i\mathrm{P}_n x}$$

we substitute the value $x = 0$ for x and obtain

$$g(0) = M\{f_1(t)\, f_2(t)\} = \sum E_n = \sum_{\varLambda_p+\mathrm{M}_q=0} A_p B_q,$$

which is the desired relation(*).

INTRODUCTORY REMARKS TO THE PROOF OF THE TWO FUNDAMENTAL THEOREMS

77. We have now arrived at the most difficult part of the theory, namely, the proof of the two (equivalent) Fundamental theorems, the uniqueness theorem and Parseval's equation. Originally, the author proved the two fundamental theorems by proving the Parseval Equation

$$M\{|f(x)|^2\} = \sum |A_n|^2$$

since he did not know at that time of the derivation of this equation from the uniqueness theorem. After the complete equivalence of the two Fundamental theorems has been shown it would naturally be convenient to give the proof on the basis of the uniqueness theorem. By what we already know this amounts to proving Parseval's equation only for the very special case where the Fourier series of the almost periodic function $f(x)$ has no terms whatsoever, and where we are dealing with the equation

$$(*) \qquad M\{|f(x)|^2\} = 0.$$

From this equation, it follows, indeed, as was shown in § 72 that $f(x)$ vanishes identically.

The equation (*), i.e., the uniqueness theorem, can be proved in several ways. Naturally one such proof results from specializing the author's original proof of the general Parseval equation. This proof, however, is very involved. Another proof, which rests on the same basic ideas was given by de la Vallée Poussin. This proof will be given here. In order that the ingenious and rather subtle reasoning of the proof may stand out as clearly as possible, we shall first briefly sketch the starting point of the original proof. We shall then have the opportunity of seeing where the principle simplifications of the new proof lie.

78. Let an almost periodic function $f(x)$ be given for which

$$a(\lambda) = M\{f(x)\, e^{-i\lambda x}\} = 0$$

for each λ. Then we are to show that

$$M\{|f(x)|^2\} = \lim_{T\to\infty} \frac{1}{T}\int_0^T |f(x)|^2\,dx = 0.$$

Since the assumption $a(\lambda) = 0$ is put to use only at the conclusion, it will be convenient not to make this assumption for the time being. Our considerations refer, then, to an arbitrary almost periodic function $f(x)$.

The idea of the proof consists in considering the purely periodic function $F(x) = F_T(x)$ with period T

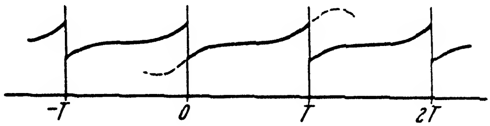

Fig. 8.

which is equal to the given function $f(x)$ in the interval $0 < x < T$. Thus

$$F_T(x) = f(x) \quad \text{for} \quad 0 < x < T, \qquad F_T(x+T) = F_T(x).$$

We then apply Parseval's equation to the function. The function $F_T(x)$ need not be continuous, but will, in general, have a saltus at each of the points mT $(m = 0, \pm 1, \pm 2, \ldots)$. This does not matter for we have considered functions of this type before, in §§ 37–38, in anticipation, in fact, of just this application. We have shown, among other results, that Parseval's equation is valid for these functions. Let us write the Fourier series of $F_T(x)$ in the form

$$F_T(x) \sim \sum_{-\infty}^{\infty} \alpha_n e^{i\frac{2\pi}{T}nx}.$$

(Here not only the exponents $\frac{2\pi}{T}n$, but also the coefficients

$$\alpha_n = \frac{1}{T}\int_0^T F_T(x)\,e^{-i\frac{2\pi}{T}nx}\,dx = \frac{1}{T}\int_0^T f(x)\,e^{-i\frac{2\pi}{T}nx}\,dx$$

depend on T). There follows the relation

$$\sum_{-\infty}^{\infty} |\alpha_n|^2 = \frac{1}{T}\int_0^T |F_T(x)|^2\,dx = \frac{1}{T}\int_0^T |f(x)|^2\,dx.$$

Now in this relation the right hand member approaches $M\{|f(x)|^2\}$; as $T \to \infty$, and therefore the same is true of the left hand member; i.e., for the mean value $M\{|f(x)|^2\}$ we now have the new expression

$$M\{|f(x)|^2\} = \lim_{T\to\infty} \sum_{-\infty}^{\infty} |\alpha_n|^2.$$

We have been considering an arbitrary almost periodic function; now we can go back to the original problem. This problem can be expressed as follows: We are to show that, from the assumption $a(\lambda) = 0$, the limit equation

$$\lim_{T\to\infty} \sum_{-\infty}^{\infty} |\alpha_n|^2 = 0.$$

follows or, in other words, that for large T the sum of squares $\sum_{-\infty}^{\infty} |\alpha_n|^2$ will turn out to be very small.

79. This is this way the uniqueness theorem can be proved. The direct proof of the given limit equation is, however, very difficult. On the other hand, as we shall very soon show, it is not particularly difficult to show that for large T the upper bound of the members $|\alpha_n|$ is very small, i.e., that from the assumption $a(\lambda) = 0$ there follows that

$$(*) \qquad \lim_{T\to\infty} \underset{-\infty<n<\infty}{\text{upper bound}} |\alpha_n| = 0 .$$

It is very interesting to realize that this much weaker limit equation yields the uniqueness theorem. To do this, however, we must alter the starting point of the proof as did de la Vallée Poussin. We must compare not the two functions $f(x)$ and $F_T(x)$, but the two folded functions

$$g(t) = \underset{t}{M}\{f(x+t)\,\overline{f(t)}\} \qquad \text{and} \qquad G_T(x) = \frac{1}{T}\int_0^T F_T(x+t)\,\overline{F_T(t)}\,dt.$$

Before examining this matter in more detail, we first give the proof of the limit equation (*).

PRELIMINARIES FOR THE PROOF OF THE UNIQUENESS THEOREM

80. First we wish to derive some lemmas.

THEOREM I. Let $f(x)$ be an arbitrary almost periodic function and ε an arbitrary positive quantity. Then numb-

ers Λ and T_0 can be chosen so large that for each $|\lambda| > \Lambda$ and each $T > T_0$

$$\left|\frac{1}{T}\int_0^T f(x)\, e^{-i\lambda x}\, dx\right| < \varepsilon .$$

Proof: From what precedes, it is clear that we can so choose the constants that for $|\lambda| > \Lambda$ this inequality holds:

$$|a(\lambda)| = \left|\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x)\, e^{-i\lambda x}\, dx\right| < \varepsilon .$$

For, there exists for a given $\varepsilon > 0$ only a finite number of numbers λ for which $|a(\lambda)| \geqq \varepsilon$. From this the inequality in question follows but with T_0 dependent on λ. To show that T_0 can be chosen independently of λ, we must proceed differently. This proof is based on the uniform continuity and boundedness of $f(x)$. In fact we even could choose $T_0 = 1$.

First we notice that for a given $\lambda \neq 0$ the function $e^{-i\lambda x}$ is periodic with period $2\pi/|\lambda|$, and that, further, the integral of this function over an arbitrary interval of length $2\pi/|\lambda|$ equals zero. From this we get for any x_1, the equation

$$\int_{x_1}^{x_1+\frac{2\pi}{|\lambda|}} f(x)\, e^{-i\lambda x}\, dx = \int_{x_1}^{x_1+\frac{2\pi}{|\lambda|}} (f(x) - f(x_1))\, e^{-i\lambda x}\, dx .$$

We let $\omega(\delta)$ denote the upper limit of $|f(x_1) - f(x_2)|$ for $|x_1 - x_2| \leqq \delta$, for a given $\delta > 0$, then we have

$$\left|\int_{x_1}^{x_1+\frac{2\pi}{|\lambda|}} f(x)\, e^{-i\lambda x}\, dx\right| \leqq \omega\left(\frac{2\pi}{|\lambda|}\right)\cdot\frac{2\pi}{|\lambda|} .$$

Let an arbitrary $T > 0$ be given and let $T = n\,\frac{2\pi}{|\lambda|} + \alpha$, where n is an integer and $0 \leqq \alpha < \frac{2\pi}{|\lambda|}$. Then

$$\int_0^T f(x)\, e^{-i\lambda x}\, dx = \int_0^{\frac{2\pi}{|\lambda|}} + \int_{\frac{2\pi}{|\lambda|}}^{\frac{4\pi}{|\lambda|}} + \cdots + \int_{(n-1)\frac{2\pi}{|\lambda|}}^{n\frac{2\pi}{|\lambda|}} + \int_{n\frac{2\pi}{|\lambda|}}^{T} f(x)\, e^{-i\lambda x}\, dx ,$$

and thus, letting Γ denote the upper bound of $|f(x)|$, we have

$$\left|\int_0^T f(x)\, e^{-i\lambda x}\, dx\right| \leqq n\,\omega\left(\frac{2\pi}{|\lambda|}\right)\frac{2\pi}{|\lambda|} + \Gamma\alpha \leqq T\cdot\omega\left(\frac{2\pi}{|\lambda|}\right) + \Gamma\frac{2\pi}{|\lambda|}.$$

Thus we find finally that for $T > T_0 = 1$

$$\left|\frac{1}{T}\int_0^T f(x)\, e^{-i\lambda x}\, dx\right| \leqq \omega\left(\frac{2\pi}{|\lambda|}\right) + \frac{\Gamma\cdot\frac{2\pi}{|\lambda|}}{T} \leqq \omega\left(\frac{2\pi}{|\lambda|}\right) + \Gamma\frac{2\pi}{|\lambda|}.$$

From here on the proof is obvious. For from the uniform continuity of $f(x)$, it follows that $\omega(\delta) \to 0$ as $\delta \to 0$. Thus a $\Lambda = \Lambda(\varepsilon)$ can so be chosen that for $|\lambda| > \Lambda$ the following inequality holds:

$$\omega\left(\frac{2\pi}{|\lambda|}\right) + \Gamma\frac{2\pi}{|\lambda|} < \varepsilon.$$

THEOREM II. Let $f(x)$ be an almost periodic function and let λ_0 be a real number for which $a(\lambda_0) = M\{f(x)\,e^{-i\lambda_0 x}\} = 0$ (i.e., λ_0 is not one of the fourier exponents of $f(x)$,) · then for each given $\varepsilon > 0$ the positive numbers δ and T_0 can be determined so that for each λ of the interval $(\lambda_0 - \delta,\ \lambda_0 + \delta)$ and each $T > T_0$

$$\left|\frac{1}{T}\int_0^T f(x)\, e^{-i\lambda x}\, dx\right| < \varepsilon.$$

Proof: It is clear from what precedes, that we can choose the constant δ so that for $|\lambda - \lambda_0| < \delta$ we have the equation

$$|a(\lambda)| = \left|\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x)\, e^{-i\lambda x}\, dx\right| < \varepsilon.$$

From this the equation in question follows except that again T_0 depends on λ. To show that T_0 can be chosen independently of λ, we must proceed differently. The proof depends on the improved mean value theorem for almost periodic functions.

We notice first, that without loss of generality we can

assume that $\lambda_0 = 0$. (For otherwise we need only consider the function $f_1(x) = f(x)\,e^{-i\lambda_0 x}$ in place of $f(x)$. On hypothesis is, then, that

$$a(0) = M\{f(x)\} = 0,$$

and we have to show that

$$\left|\frac{1}{T}\int_0^T f(x)\,e^{-i\lambda x}\,dx\right| < \varepsilon \text{ for } |\lambda| < \delta, \quad T > T_0.$$

In this formulation the theorem has a marked similarity to Theorem I, but there we were concerned with the very "rapid" oscillation $e^{-i\lambda x}$ (namely with the large values of $|\lambda|$ and the proof rested on the fact that factor $f(x)$ hardly changes in the course of a full oscillation of $e^{-i\lambda x}$. In the theorem to be proved now the situation is exactly reversed; here we are concerned with very "slow" oscillations $e^{-i\lambda x}$ (where the values of $|\lambda|$ are small, and the proof will depend on the fact that the factor $e^{-i\lambda x}$ changes too slowly to cancel out the oscillations of $f(x)$ that are implied in the condition $M\{f(x)\} = 0$. The idea is carried out as follows:

By the improved mean value theorem of § 52, the mean value

$$\frac{1}{T}\int_a^{a+T} f(x)\,dx$$

converges to $M\{f(x)\}$, uniformly in a, as $T \to \infty$. Thus in this case, it converges to zero and we can first choose T_0 so large that for $H > T_0$ and each a

$$\left|\frac{1}{H}\int_a^{a+H} f(x)\,dx\right| < \frac{\varepsilon}{2}.$$

This T_0 will be the T_0 of our theorem. Let Γ now be the upper limit of $|f(x)|$ and let us now choose the number $\eta > 0$ so small that

$$\underset{|x_1 - x_2| < \eta}{\text{upper limit}} \; |e^{ix_1} - e^{ix_2}| < \frac{\varepsilon}{2\Gamma}$$

(which is possible because of the uniform continuity of e^{ix}). If we choose $\delta = \eta/2T_0$ then for each $|\lambda| < \delta$.

$$\underset{|x_1 - x_2| < 2T_0}{\text{upper limit}} \left| e^{-i\lambda x_1} - e^{-i\lambda x_2} \right| < \frac{\varepsilon}{2\Gamma} .$$

This δ will be the δ of our theorem. Now it is always true that

$$\frac{1}{H}\int_a^{a+H} f(x)\, e^{-i\lambda x}\, dx = \frac{1}{H}\int_a^{a+H} f(x)\, e^{-i\lambda a}\, dx + \frac{1}{H}\int_a^{a+H} f(x)\left(e^{-i\lambda x} - e^{-i\lambda a}\right) dx.$$

Thus it follows from the above that for $|\lambda| < \delta$ for $T_0 < H \leqq 2T_0$ and, for each a the following inequality holds:

$$\left|\frac{1}{H}\int_a^{a+H} f(x)\, e^{-i\lambda x}\, dx\right| < \frac{\varepsilon}{2}\cdot\left|e^{-i\lambda a}\right| + \Gamma\cdot\frac{\varepsilon}{2\Gamma} = \varepsilon .$$

The proof of our theorem is now nearly complete. Let T be an arbitrary number $> T_0$; we can then choose a posi-

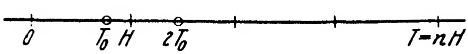

Fig. 9.

tive integer n such that the number $T/n = H$ lies in the interval $T_0 < H \leqq 2T_0$. (We have only to determine that integer $k \geqq 0$ for which $2^k T_0 < T \leqq 2^{k+1} T_0$, and then the number in question can be chosen as $n = 2^k$). Then the mean value

$$\frac{1}{T}\int_0^T f(x)\, e^{-i\lambda x}\, dx$$

is the arithmetic mean of the n mean value

$$\frac{1}{H}\int_{(\nu-1)H}^{\nu H} f(x)\, e^{-i\lambda x}\, dx\,, \qquad \nu = 1, 2, \ldots, n$$

and it is, therefore, numerically smaller than ε for $|\lambda| < \delta$.

81. With the help of Theorems I and II, we now prove the following:

PRINCIPAL LEMMA. <u>Let</u> $f(x)$ <u>be an almost periodic function for which</u> $a(\lambda) = M\{f(x)e^{-i\lambda x}\} = 0$ <u>for each and every</u> λ. <u>Then the limit equation</u>

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)\, e^{-i\lambda x}\, dx = 0$$

holds uniformly for $-\infty < \lambda < \infty$, i.e., to a given $\varepsilon > 0$ a $T_0 = T_0(\varepsilon)$ can be chosen so that for $T > T_0$ and all λ

$$\left|\frac{1}{T}\int_0^T f(x)\, e^{-i\lambda x}\, dx\right| < \varepsilon .$$

Proof: We first choose the number Λ according to Theorem I to be so large that for $|\lambda| > \Lambda$ and each $T > 1$

$$\left|\frac{1}{T}\int_0^T f(x)\, e^{-i\lambda x}\, dx\right| < \varepsilon .$$

Thus, according to Theorem II we determine a positive number $\delta = \delta(\lambda_0)$ and $T_0 = T_0(\lambda_0)$, corresponding to an arbitrary fixed λ_0 in the interval $-\Lambda \leqq \lambda_0 \leqq \Lambda$ such that for each λ of the interval $(\lambda_0 - \delta(\lambda_0), \lambda_0 + \delta(\lambda_0))$ and each $T > T_0(\lambda_0)$

$$\left|\frac{1}{T}\int_0^T f(x)\, e^{-i\lambda x}\, dx\right| < \varepsilon .$$

The interval $(\lambda_0 - \delta(\lambda_0), \lambda_0 + \delta(\lambda_0))$ will be denoted for brevity by $i(\lambda_0)$. Obviously because of the method of determination of this interval $i(\lambda_0)$, the conditions are satisfied for the application of the Heine-Borel covering theorem; For each point λ_0 of the closed interval $-\Lambda \leqq \lambda \leqq \Lambda$ is contained in $i(\lambda_0)$ (as its midpoint). By the Heine-Borel theorem it is possible to select a finite number of points of the interval $-\Lambda \leqq \lambda \leqq \Lambda$ e.g., $\lambda_1, \lambda_2, \ldots, \lambda_N$, such that the corresponding intervals $i(\lambda_1), i(\lambda_2), \ldots, i(\lambda_N)$ cover the complete interval $-\Lambda \leqq \lambda \leqq \Lambda$. Then the number

$$T_0 = \operatorname{Max}(1, T_0(\lambda_1), T_0(\lambda_2), \ldots, T_0(\lambda_N))$$

has the desired property. In fact if λ is a completely arbitrary real number then it will either be outside the interval $-\Lambda \leqq \lambda \leqq \Lambda$, (i.e., satisfy the inequality $|\lambda| > \Lambda$) or it will belong to at least one of the intervals $i(\lambda_1), i(\lambda_2), \ldots i(\lambda_N)$ and thus in either case the following inequality is valid:.

$$\left|\frac{1}{T}\int_0^T f(x)\,e^{-i\lambda x}\,dx\right| < \varepsilon \text{ for } T > T_0.$$

82. To bring this result into the handiest form for the following application, we consider the purely periodic function introduced above, namely $F_T(x)$ of period T, which coincides with $f(x)$ in the interval $0 < x < T$. Let the ordinary Fourier series of the function $F_T(x)$, be $\sum_{-\infty}^{\infty} \alpha_n e^{i\frac{2\pi}{T}nx}$ i.e., let

$$\alpha_n = \frac{1}{T}\int_0^T F_T(x)\,e^{-i\frac{2\pi}{T}nx}\,dx = \frac{1}{T}\int_0^T f(x)\,e^{-i\frac{2\pi}{T}nx}\,dx.$$

From the above the following theorem follows immediately. For each given $\varepsilon > 0$, we can determine a $T_0 = T_0(\varepsilon)$ such that for $T > T_0$ all Fourier constants α_n of $F_T(x)$ satisfy the inequality $|\alpha_n| < \varepsilon$; or in other words: <u>If the condition $a(\lambda) = 0$ is true for all λ for the function $f(x)$, then the limit equation</u>

$$\lim_{T\to\infty} \underset{\infty < n < \infty}{\text{upper bound}} |\alpha_n| = 0.$$

<u>is true</u>.

Thus we have proved the relation (*) of § 79 on which the de la Vallée Poussin proof of the uniqueness theorem is based. From this limit equation, however, we can not conclude that

$$\lim_{T\to\infty} \sum_{-\infty}^{\infty} |\alpha_n|^2 = 0$$

i.e., that $\sum_{-\infty}^{\infty}|\alpha_n|^2$ is small for large T, which would be rather convenient for a direct proof of the uniqueness theorem using the function $F_T(x)$. But we use the simple fact that $\sum_{-\infty}^{\infty}|\alpha_n|^2$ is always <u>bounded</u>, or

$$\sum_{-\infty}^{\infty}|\alpha_n|^2 = \frac{1}{T}\int_0^T |F_T(x)|^2\,dx = \frac{1}{T}\int_0^T |f(x)|^2\,dx \leq \Gamma^2,$$

(where Γ as always, denotes the upper limit of $|f(x)|$ in $-\infty < x < \infty$). Thus we can conclude that, for example, the sum of the <u>fourth</u> powers $\sum_{-\infty}^{\infty}|\alpha_n|^4$ is small for large values of T.

In fact, we verify that

$$\sum_{-\infty}^{\infty}|\alpha_n|^4 \leqq (\underset{-\infty<n<\infty}{\text{upper limit}}|\alpha_n|^2)\cdot\sum_{-\infty}^{\infty}|\alpha_n|^2 \leqq (\text{upper limit}|\alpha_n|)^2\cdot\Gamma^2,$$

where the last quantity approaches zero as $T\to\infty$. In other words the following theorem is true: <u>If for the function $f(x)$ the condition $a(\lambda)=0$ is satisfied for all λ, then</u>

$$\lim_{T\to\infty}\sum_{-\infty}^{\infty}|\alpha_n|^4 = 0.$$

On this theorem we will base the following proof of the uniqueness theorem.

PROOF OF THE UNIQUENESS THEOREM

83. Let $f(x)$ be an almost periodic function, about which we shall at first make no assumption. We introduce the function $F_T(x)$ as above and consider the two folded functions:

$$g(x) = \underset{t}{M}\{f(x+t)\,\overline{f(t)}\} \quad \text{and} \quad G_T(x) = \frac{1}{T}\int_0^T F_T(x+t)\,\overline{F_T(t)}\,dt.$$

Here, as we know, $g(x)$ is also an almost periodic function and $G_T(x)$ is a (continuous) purely periodic function of period T, whose Fourier series is given by

$$G_T(x) \sim \sum|\alpha_n|^2 e^{i\frac{2\pi}{T}nx}.$$

From this it follows that for each T

$$\frac{1}{T}\int_0^T|G_T(x)|^2\,dx = \sum_{-\infty}^{\infty}|\alpha_n|^4.$$

The function $G_T(x)$ naturally does not need to coincide with the function $g(x)$ in the period-interval $0<x<T$. Indeed the function does not depend in the same way on $g(x)$ as $F_T(x)$ depends on $f(x)$. This is, however, not decisive; it will be sufficient if $G_T(x)$ provides a "good approximation" to $g(x)$ in the interval $0<x<T$ but even this is, in general, not the case for arbitrary values of T. As we know from § 53, this limit equation

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x+t)\,\overline{f(t)}\,dt = \underset{t}{M}\{f(x+t)\,\overline{f(t)}\}$$

holds uniformly in x as $T\to\infty$. A comparison of the two functions $g(x)$ and $G_T(x)$ in the interval $0<x<T$ amounts in essence to a comparison of the two functions

$$g_T(x)=\frac{1}{T}\int_0^T f(x+t)\,\overline{f(t)}\,dt$$

and

$$G_T(x)=\frac{1}{T}\int_0^T F_T(x+t)\,\overline{F_T(t)}\,dt\ .$$

The two functions $g_T(x)$ and $G_T(x)$ are now by no means identical in $0<x<T$. The equations $f(t)=F_T(t)$ and $\overline{f(t)}=\overline{F_T(t)}$, indeed, hold for the entire interval of integration $0<t<T$ but the other factors $f(x+t)$ and $F_T(x+t)$ do not coincide in the entire interval $0<t<T$ since $x+t$ lies partially in the interval $T<x+t<2T$ for $0<t<T$.

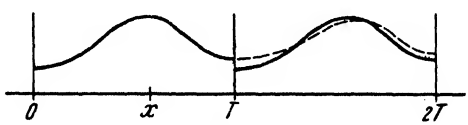

Fig. 10.

We now use our trick, which consists of taking for the "slice" T not an arbitrary number but a translation number say $T=\tau_f(\varepsilon)$. Then for $T<z<2T$

$$|f(z)-F_T(z)|=|f(z)-f(z-T)|\leqq\varepsilon,$$

and from that the inequality

$$|f(x+t)-F_T(x+t)|\leqq\varepsilon.$$

holds for each fixed x in $0<x<T$, and for the entire interval $0<t<T$.

Thus we obtain (with $T=\tau_f(\varepsilon)$) the following inequality, valid for all x in $0<x<T$

$$|g_T(x)-G_T(x)|\leqq\varepsilon\cdot\Gamma.$$

Let a sequence of numbers $\varepsilon_1>\varepsilon_2>\cdots>\varepsilon_n>\cdots\to 0$ be arbitrarily chosen (e.g. $\varepsilon_n=1/n$) and let $T_1, T_2,\ldots,T_n,\ldots$ be a sequence of corresponding translation numbers (i.e., $\Gamma_n=\tau_f(\varepsilon_n)$) for which $T_n\to\infty$. Then for each value of n,

$$|g_{T_n}(x)-G_{T_n}(x)|\leqq\varepsilon_n\Gamma\quad\text{in}\quad 0<x<T_n,$$

and thus,

$$\text{upper limit}_{0<x<T_n} |g_{T_n}(x) - G_{T_n}(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

Now, however, the sequence of functions $g_{T_n}(x)$ converges to $g(x)$ for $n \to \infty$ uniformly in $-\infty < x < \infty$. In particular it is true that

$$\text{upper limit}_{0<x<T_n} |g(x) - g_{T_n}(x)| \to 0 \quad \text{as} \quad n \to \infty$$

Thus we finally obtain as a result of the comparison of the functions $g(x)$ and $G_T(x)$, the result that <u>there exists a sequence of numbers</u> $T_1, T_2, \ldots, T_n, \ldots$ <u>of some kind with</u> $T_n \to \infty$, <u>such that</u>

$$\text{upper bound}_{0<x<T_n} |g(x) - G_{T_n}(x)| \to 0 \quad \text{as} \quad n \to \infty;$$

In other words, the function $G_{T_n}(x)$ represents in the interval $0 < x < T_n$ an ever better approximation of $g(x)$ as n becomes infinite.

From this we can conclude without difficulty that

$$(*) \qquad M\{|g(x)|^2\} = \lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} |G_{T_n}(x)|^2\,dx \quad .$$

For it is true that for a given n

$$\big||g(x)|^2 - |G_{T_n}(x)|^2\big| = \big||g(x)| - |G_{T_n}(x)|\big| \cdot (|g(x)| + |G_{T_n}(x)|)$$
$$\leqq \text{upper limit}_{0<x<T_n} |g(x) - G_{T_n}(x)| \cdot 2\Gamma^2$$

in the interval $0 < x < T_n$.

Thus,

$$\left|\frac{1}{T_n}\int_0^{T_n} |g(x)|^2\,dx - \frac{1}{T_n}\int_0^{T_n} |G_{T_n}(x)|^2\,dx\right| \leqq \frac{1}{T_n}\int_0^{T_n} \big||g(x)|^2 - |G_{T_n}(x)|^2\big|\,dx$$
$$\leqq \text{upper limit}_{0<x<T_n} |g(x) - G_{T_n}(x)| \cdot 2\Gamma^2.$$

From this, the above limit-equation (*) follows; for the last quantity approaches zero as $n \to \infty$ and further, since the quantity $\frac{1}{T_n}\int_0^{T_n} |g(x)|^2\,dx$ converges to the mean value $M\{|g(x)|^2\}$, the quantity $\frac{1}{T_n}\int_0^{T_n} |G_{T_n}(x)|^2\,dx$. must approach this mean value $M\{|g(x)|^2\}$.

After this preparation it is easy to give the proof of

the uniqueness theorem. Let the hypothesis

$$a(\lambda) = M\{f(x)\,e^{-i\lambda x}\} = 0$$

be satisfied for $f(x)$ and all λ. We are to prove that $f(x)$ vanishes identically.

First, the assumption $a(\lambda) = 0$ for all λ implies, as shown in § 82, that

$$\lim_{T\to\infty} \sum_{-\infty}^{\infty} |\alpha_n|^4 = 0 \ .$$

Since now,

$$\frac{1}{T}\int_0^T |G_T(x)|^2\,dx = \sum_{-\infty}^{\infty}{}' |\alpha_n|^4$$

for every T it follows that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |G_T(x)|^2\,dx = 0 \ .$$

With this, after what has preceded, we must conclude that

$$M\{|g(x)|^2\} = \lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} |G_{T_n}(x)|^2\,dx = 0 \,.$$

From the property of the almost periodic function $g(x)$ expressed in $M\{|g(x)|^2\} = 0$ it follows, however, that $g(x)$ vanishes identically. Thus it follows, in particular, that

$$g(0) = \underset{t}{M}\{f(t)\,\overline{f(t)}\} = M\{|f(t)|^2\} = 0 \ .$$

From the equation $M\{|f(t)|^2\} = 0$ it further follows that $f(x)$ itself must vanish identically, which completes the proof.

THE FUNDAMENTAL THEOREM

84. Now that the uniqueness theorem, and with it, Parseval's equation are demonstrated, we turn to the proof of the real Fundamental Theorem of the theory of almost periodic functions: The closure $H\{s(x)\}$ of the class of all finite sums $s(x) = \sum_1^N a_n e^{i\lambda_n x}$ is identical with the class of almost periodic functions.

We already know (§ 49) that the class $H\{s(x)\}$ is con-

tained in the class of almost periodic functions. It remains to be proven that conversely the class of almost periodic functions is contained in the class $H\{s(x)\}$, i.e., that the following theorem holds.

APPROXIMATION THEOREM. Every almost periodic function $f(x)$ can be approximated by finite sums $s(x)=\sum_1^N a_n e^{i\lambda_n x}$ uniformly for $-\infty<x<\infty$ i.e., for each $\varepsilon>0$, there exists a sum $s(x)$ of such a nature that

$$|f(x)-s(x)|\leqq\varepsilon \text{ for all } x.$$

The exponents in the approximating sums $s(x)$ can be chosen to be precisely the Fourier exponents Λ_n of the function $f(x)$.

It suffices in what follows to consider the case where the Fourier series of $f(x)$ contains infinitely many terms, since, otherwise, by the uniqueness theorem, the function would itself be a finite sum $s(x)$.

If any finite sum $s(x)$ approximates the function $f(x)$ with the degree of accuracy ε i.e., $|f(x)-s(x)|\leqq\varepsilon$ for all x, then it is clear that a "foreign" exponent (i.e., one which is distinct from the Fourier coefficients Λ_n) can enter only in connection with a coefficient of absolute value $\leqq\varepsilon$ since

$$|a_n|=|M\{s(x)\,e^{-i\lambda_n x}\}|=|M\{(s(x)-f(x))\,e^{-i\lambda_n x}\}|\leqq M\{\varepsilon\cdot 1\}=\varepsilon.$$

For various applications of the approximation theorem it is of the greatest importance that one can completely omit such "foreign" exponents λ_n just as in the case of purely periodic functions.

On the other hand for a sufficiently fine approximation every preassigned Fourier exponent Λ_n must as a matter of fact enter into the approximating sum $s(x)$; to be exact, whenever the measure of accuracy ε, is less than $|A_n|$.

85. In all our previous considerations of the Fourier series of an almost periodic function there was no reason for choosing any particular order of terms. As in the corresponding place in the theory of Fourier series of purely periodic functions, for the proof of the approximation theorem, we have first to discuss in detail the

question of the ordering of the terms. We shall find it closest to classical Fourier theory to consider only the Fourier exponents Λ_n (and not the coefficient A_n), in the ordering.

Since a completely arbitrary enumerable set of real numbers can occur as the set of exponents $\{\Lambda_n\}$, would at first appear to be hopeless to give a general principle that, when applied to any almost periodic function, will systematically exhaust its set of exponents. By recourse to an arithmetical point of view, however, we can do this very simply by introducing the concept of a "basis" of the set of real numbers $\{\Lambda_n\}$.

In the case of a finite set of numbers $\Lambda_1, \ldots, \Lambda_N$, the concept of a basis is a very familiar one, and indeed one understands by it a finite set of numbers $\beta_1, \ldots, \beta_M$ of such a nature that each of the given numbers Λ_n can be expressed in only one way in the form

$$\Lambda_n = g_{n,1}\beta_1 + \cdots + g_{n,M}\beta_M$$

with integral coefficients $g_{n,m}$. In the transition to an enumerable set $\Lambda_1, \Lambda_2, \ldots$ the concept of a basis must be extended since the basis must include enumerably many numbers $\beta_1, \beta_2, \ldots$ In order to be able to introduce the concept of a basis of an arbitrary enumerable set of numbers, it is necessary to permit the coefficients $g_{n,m}$ to be not only integral but also rational.

86. Let $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ be any set of enumerably many real numbers. By a basis

$$\mathrm{B} = \{\beta_1, \beta_2, \beta_3, \ldots\}$$

we mean an at most enumerable set of real numbers β_n with the following properties:

1. The numbers $\beta_1, \beta_2, \beta_3, \ldots$ are linearly independent, i.e., no relation holds among any m of them of the following form:

$$R_1\beta_1 + R_2\beta_2 + \cdots + R_m\beta_m = 0$$

with rational coefficients $R_1, R_2, \ldots, R_m$ which do not all vanish.

2. Each of the numbers Λ_n can be represented, for sufficiently large m in the form

$$\Lambda_n = r_1\beta_1 + r_2\beta_2 + \cdots + r_m\beta_m$$

with rational coefficients $r_1, r_2, \ldots, r_m$. Here it is obvious from 1., that the representation of Λ_n is unique (except, of course, for terms with zero coefficients).

<u>Example</u>: The set of all rational numbers $\{\Lambda_n\}$ has the basis $\{1\}$ The set $\Lambda_n = \log n$ $(n = 1, 2, 3, \ldots)$ has as basis the set $\{\log p_n\}$, where p_n runs through the sequence of prime numbers $2, 3, 5, \ldots$

87. We now wish to show that <u>each sequence</u> $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ <u>has a basis</u> and indeed one whose numbers β_n all belong to the set $\{\Lambda_n\}$. Here we proceed as follows:

As the first number of our basis β_1 we take the first number Λ_{n_1} of the sequence $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$, which is not zero (thus either Λ_1 or Λ_2), and we cross out each number Λ_n of the sequence $\Lambda_{n_1+1}, \Lambda_{n_1+2}, \ldots$, for which a relation of the form $r_1\beta_1 + r_2\Lambda_n = 0$ holds in rational numbers r_1, r_2 with $r_2 \neq 0$. Let Λ_{n_2} be the first number of the sequence $\Lambda_{n_1+1}, \Lambda_{n_1+2}, \ldots$, which is not crossed out in this manner. Then we set $\beta_2 = \Lambda_{n_2}$ and cross out every number Λ_n still remaining in the sequence $\Lambda_{n_2+1}, \Lambda_{n_2+2}, \ldots$, for which a relation holds of the form $r_1\beta_1 + r_2\beta_2 + r_3\Lambda_n = 0$ in rational numbers r_1, r_2, r_3 with $r_3 \neq 0$. After that we set $\beta_3 = \Lambda_{n_3}$ where Λ_{n_3} denotes the first number remaining in the sequence $\Lambda_{n_2+1}, \Lambda_{n_2+2}, \ldots$ etc. Now there are two distinct possibilities. On one hand, this procedure can terminate after a finite number of steps, say p steps, so that the entire set of numbers $\Lambda_{n_p+1}, \Lambda_{n_p+2}, \ldots$ are crossed out (and thus no Λ remain). In this case the finite set

$$\{\beta_1, \beta_2, \ldots, \beta_p\} = \{\Lambda_{n_1}, \Lambda_{n_2}, \ldots, \Lambda_{n_p}\}$$

obviously forms a basis. On the other hand the procedure may never terminate. In this case a basis will obviously be given by the enumerable sequence

$$\{\beta_1, \beta_2, \beta_3, \ldots\} = \{\Lambda_{n_1}, \Lambda_{n_2}, \Lambda_{n_3}, \ldots\}.$$

88. We choose <u>any basis at all</u> $B = \{\beta_1, \beta_2, \beta_3, \ldots\}$ <u>for the sequence of exponents</u> $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ <u>of our given almost periodic function</u> $f(x)$. <u>From now on this basis will be consid-</u>

ered fixed. For the sake of convenience we wish to assume that each number of the enumerable set of numbers

$$r_1\beta_1 + r_2\beta_2 + \cdots + r_m\beta_m$$

occurs among the exponents Λ_n (where m is arbitrary and $r_1, r_2, \ldots, r_m$ are arbitrary rational numbers). We have only to add the missing terms $A_n e^{i\Lambda_n x}$ with coefficients $A_n = 0$ in the series $\sum A_n e^{i\Lambda_n x}$. We carry out the proof of the approximation theorem for the case where the basis $\{\beta_1, \beta_2, \beta_3, \ldots\}$ is enumerably infinite. The simplifications that are intended when our basis is finite are evident. (We could immediately go from this case to the general case by successive addition of new numbers to extend the basis to an infinite basis).

Now let q be an arbitrary positive integer and let Q denote $q! = 1 \cdot 2 \cdot 3 \cdots q$. Further let P denote the number $P = qQ (= q \cdot q!)$ so that $\frac{P}{Q} \to \infty$ as $q \to \infty$. We denote by E_q the finite set which consists of all numbers Λ_n which can be written in the form

$$\Lambda_n = \frac{\nu_1}{Q}\beta_1 + \frac{\nu_2}{Q}\beta_2 + \cdots + \frac{\nu_q}{Q}\beta_q$$

with the help of our basis, when $\nu_1, \nu_2, \ldots, \nu_q$ denote integers all numerically $\leqq P (= qQ)$. We write an arbitrary number Λ_n in the form

$$\Lambda_n = r_1\beta_1 + r_2\beta_2 + \cdots + r_m\beta_m = \frac{s_1}{t_1}\beta_1 + \frac{s_2}{t_2}\beta_2 + \cdots + \frac{s_m}{t_m}\beta_m,$$

where we assume that the fractions $s_1/t_1, s_2/t_2, \ldots, s_m/t_m$ are reduced to their lowest terms and the denominators $t_1, t_2, \ldots, t_m$ are positive. Thus those numbers Λ_n surely belong to the set E_q in case the integers $m, t_1, t_2, \ldots, t_m$ are all $\leqq q$ and the rational numbers $r_1, r_2, \ldots, r_m$ are all numerically $\leqq q$.

Each set E_q obviously contains the preceding set E_{q-1}, i.e.,

$$E_1 \subset E_2 \subset \cdots \subset E_q \subset \cdots;$$

Further, by what we just remarked, each fixed

$$\Lambda_n = r_1\beta_1 + r_2\beta_2 + \cdots + r_m\beta_m = \frac{s_1}{t_1}\beta_1 + \frac{s_2}{t_2}\beta_2 + \cdots + \frac{s_m}{t_m}\beta_m$$

is surely contained in some set E_q for sufficiently large q, i.e., for $q \geqq q_0 = q_0(n)$, where by what precedes, we can use for q_0 the value

$$q_0 = \mathrm{Max}\{m, t_1, t_2, \ldots, t_m, |r_1|, |r_2|, \ldots, |r_m|\}.$$

89. We consider now a sequence of finite sums which we write for convenience as infinite series, namely

$$s_q(x) = \sum k_n^{(q)} A_n e^{i\Lambda_n x}, \qquad q = 1, 2, 3, \ldots,$$

where the exponents Λ_n, which actually enter into the sum $s_q(x)$ (i.e., for which the factor $k_n^{(q)}$ preceding the term $A_n e^{i\Lambda_n x}$ is distinct from zero), all belong to E_q. We postulate that $0 \leqq k_n^{(q)} \leqq 1$ for all n and q, and that $k_n^{(q)} \to 1$ for n fixed and $q \to \infty$.

It is easy to show with the aid of Parseval's equation, that each such sequence of finite sums $s_q(x)$ converges in the mean to $f(x)$, i.e., that

$$M\{|f(x) - s_q(x)|^2\} \to 0 \quad \text{as} \quad q \to \infty.$$

In fact from $f(x) \sim \sum A_n e^{i\Lambda_n x}$ and $s_q(x) = \sum k_n^{(q)} A_n e^{i\Lambda_n x}$, it follows that

$$f(x) - s_q(x) \sim \sum (1 - k_n^{(q)}) A_n e^{i\Lambda_n x}.$$

Thus it follows from Parseval's equation that

$$M\{|f(x) - s_q(x)|^2\} = \sum_1^\infty (1 - k_n^{(q)})^2 |A_n|^2.$$

Now for a given $\varepsilon > 0$ let the number N be chosen so large that $\sum_{N+1}^{\infty} |A_n|^2 < \frac{\varepsilon}{2}$, then obviously

$$M\{|f(x) - s_q(x)|^2\} < \sum_1^N (1 - k_n^{(q)})^2 |A_n|^2 + \frac{\varepsilon}{2},$$

and it follows immediately, for sufficiently large values of q, that

$$M\{|f(x) - s_q(x)|^2\} < \varepsilon.$$

For by our assumption, the N numbers $k_1^{(q)}, k_2^{(q)}, \ldots, k_N^{(q)}$ all converge to 1 as $q \to \infty$.

90. It was shown in § 73: If a sequence $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ of almost periodic functions converges <u>in the mean</u> to the almost periodic function $f(x)$ then, in order to

be able to conclude that the convergence is uniform in $-\infty < x < \infty$ it suffices to know that the sequence $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ is majorized by the function $f(x)$. The sequence of finite sums $s_q(x)$ constructed above, will, therefore, surely converge uniformly to $f(x)$ if the constants $k_n^{(q)}$ can so be chosen that the sequence of functions $s_q(x)$ is majorized by $f(x)$.

91. For this purpose we bring in again the Fejér kernel,

$$K_n(t) = \sum_{\nu=-n}^{n}\left(1 - \frac{|\nu|}{n}\right)e^{-i\nu t}$$

which played a fundamental role in the theory of purely periodic functions, and we construct, after Bochner, the "composite" kernel

$$K^{(q)}(t) = K_P\left(\frac{\beta_1}{Q}t\right)\cdot K_P\left(\frac{\beta_2}{Q}t\right)\cdots K_P\left(\frac{\beta_q}{Q}t\right)$$

$$= \sum_{\nu_1=-P}^{P}\sum_{\nu_2=-P}^{P}\cdots\sum_{\nu_q=-l}^{P}\left(1-\frac{|\nu_1|}{P}\right)\left(1-\frac{|\nu_2|}{P}\right)\cdots\left(1-\frac{|\nu_q|}{P}\right)e^{-it\left(\frac{\nu_1}{Q}\beta_1+\frac{\nu_2}{Q}\beta_2+\cdots+\frac{\nu_q}{Q}\beta_q\right)}.$$

Because of the linear independence of the numbers $\beta_1, \beta_2, \ldots, \beta_q$ no two terms of this finite sum can be combined, (i.e., no two contain the same factor $e^{i\lambda t}$). The numbers

$$\frac{\nu_1}{Q}\beta_1 + \frac{\nu_2}{Q}\beta_2 + \cdots + \frac{\nu_q}{Q}\beta_q,$$

which enter into the exponents, further, are exactly the numbers Λ_n which belong to the set E_q defined above.

We write $K^{(q)}(t)$ in the form

$$K^{(q)}(t) = \sum k_n^{(q)} e^{-i\Lambda_n t}$$

(where $k_n^{(q)} = 0$ as long as Λ_n does not belong to the set E_q). Then the numbers $k_n^{(q)}$, thus defined satisfy the above conditions $0 \leqq k_n^{(q)} \leqq 1$ for all n and q, and $k_n^{(q)} \to 1$ as $q \to \infty$. for n fixed. This is easily shown. The first is evident and the second follows thus: Let n be fixed and let

$$\Lambda_n = r_1\beta_1 + r_2\beta_2 + \cdots + r_m\beta_m = \frac{s_1}{t_1}\beta_1 + \frac{s_2}{t_2}\beta_2 + \cdots + \frac{s_m}{t_m}\beta_m.$$

Then as was noticed (in § 88) above, the number Λ_n belongs to the set E_q for $q \geqq q_0 = q_0(n)$, where

$$q_0 = \mathrm{Max}\{m, t_1, t_2, \ldots, t_m, |r_1|, |r_2|, \ldots, |r_m|\},$$

and it is obvious that

$$A_n = \frac{\nu_1}{Q}\beta_1 + \frac{\nu_2}{Q}\beta_2 + \cdots + \frac{\nu_m}{Q}\beta_m + 0\beta_{m+1} + \cdots + 0\beta_q,$$

where $\nu_1 = r_1 Q, \nu_2 = r_2 Q, \ldots, \nu_m = r_m Q$. Now we have

$$k_n^{(q)} = \left(1 - \frac{|\nu_1|}{P}\right)\left(1 - \frac{|\nu_2|}{P}\right)\ldots\left(1 - \frac{|\nu_m|}{P}\right)\cdot 1 \ldots 1,$$

and since each of the m numbers

$$\frac{|\nu_1|}{P} = |r_1|\frac{Q}{P}, \quad \frac{|\nu_2|}{P} = |r_2|\frac{Q}{P}, \quad \ldots, \quad \frac{|\nu_m|}{P} = |r_m|\frac{Q}{P}$$

approaches zero as $q \to \infty$, (since $Q/P = 1/q \to 0$ as $q \to \infty$), we have $k_n^{(q)} \to 1$ as $q \to \infty$.

92. Now we wish to show that the sequence of functions

$$s_q(x) = \sum k_n^{(q)} A_n e^{i\Lambda_n x}$$

formed with the aid of the chosen constants $k_n^{(q)}$ will be majorized by the function $f(x)$, i.e., to every arbitrarily given $\varepsilon > 0$ the corresponding translation numbers $\tau = \tau_f(\varepsilon)$ are at the same time translation numbers of each of the functions $s_q(x)$. Then by the above proof we will have shown that this sequence of finite sums $s_q(x)$ converges uniformly to $f(x)$ as $q \to \infty$.

The proof rests on the fact that for an arbitrary q

$$s_q(x) = \underset{t}{M}\{f(x+t)\, K^{(q)}(t)\},$$

where $K^{(q)}(t)$ denotes the above composite kernel. This formula is very easy to establish. In fact from

$$f(x+t) \sim \sum A_n e^{i\Lambda_n x} \cdot e^{i\Lambda_n t},$$

it follows that

$$s_q(x) = \sum k_n^{(q)} A_n e^{i\Lambda_n x} = \sum k_n^{(q)} \underset{t}{M}\{f(x+t)\, e^{-i\Lambda_n t}\}$$

$$= \underset{t}{M}\{f(x+t) \sum k_n^{(q)} e^{-i\Lambda_n t}\} = \underset{t}{M}\{f(x+t)\, K^{(q)}(t)\}.$$

Now, however, the composite kernel $K^{(q)}(t)$ possesses the same important properties as the Fejér kernel $K_n(t)$ itself; namely, first, that

$$K^{(q)}(t) \geqq 0$$

for all t, (which is evident from the definition of $K^{(q)}(t)$ as

the product of simple Fejér kernels), and secondly that

$$M\{K^{(q)}(t)\} = 1,$$

which is equally evident when one notices that the mean value $M\{K^{(q)}(t)\}$ is equal to the constant term of $K^{(q)}(t)$ which is exactly 1.

By application of these two properties of $K^{(q)}(t)$ it now follows easily that each arbitrary translation number τ of $f(x)$ corresponding to a given $\varepsilon > 0$ is at the same time a translation number of the finite sum $s_q(x)$ corresponding to ε. In fact we find, for an arbitrary x

$$\begin{aligned} s_q(x+\tau) - s_q(x) &= \underset{t}{M}\{f(x+\tau+t)\,K^{(q)}(t)\} - \underset{t}{M}\{f(x+t)\,K^{(q)}(t)\} \\ &= \underset{t}{M}\{[f(x+\tau+t) - f(x+t)]\cdot K^{(q)}(t)\}. \end{aligned}$$

Now if $$|f(x+\tau+t) - f(x+t)| \leqq \varepsilon \quad \text{for all} \quad t;$$

then it follows (since $K^{(q)}(t) \geqq 0$) that

$$|s_q(x+\tau) - s_q(x)| \leqq \varepsilon \underset{t}{M}\{K^{(q)}(t)\} = \varepsilon,$$

which shows that τ is indeed a $\tau_{s_q}(\varepsilon)$.

Thus the sequence of finite sums $s_q(x)$ converges uniformly to $f(x)$, as $q \to \infty$ and the proof of the fundamental theorem is complete.

AN IMPORTANT EXAMPLE

93. As a special case of almost periodic functions we have previously considered thus far only the purely periodic functions of period 2π. These functions are characterized (within the class of almost periodic functions) by the fact that their Fourier exponents are all integral; the exponents are, therefore, in this case rather strongly connected arithmetically -- which gave rise to the expression that the oscillations e^{inx} are harmonic.

In contrast to this we wish now to consider the case of completely disharmonic oscillations, i.e., the case when the exponents Λ_n are linearly independent, i.e., where there exists no linear relation whatsoever

$$g_1\Lambda_1 + g_2\Lambda_2 + \cdots + g_m\Lambda_m = 0$$

with integral non zero coefficients $g_1, g_2, \ldots, g_m$. In other words the set of exponents $\{\Lambda_n\}$ forms a basis here. (In particular no Λ_n is 0).

This case can in a certain sense, be thought of as the "general case", if one thinks of the Fourier exponents as being chosen in succession in a purely random or chance manner. In fact if the first n Fourier coefficients $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ are fixed there are only enumerably many numbers linearly dependent on these, while there is a non-enumerable set of real numbers λ available from which to choose Λ_{n+1}

In this case of linear independence, the results are as simple as one can imagine. In particular the following theorem is valid:

CONVERGENCE THEOREM: For an almost periodic function $f(x)$ with linearly independent Fourier exponents the Fourier series $\sum A_n e^{i\Lambda_n x}$ (with arbitrary arrangement of terms) converges uniformly for all x, since, in fact, the series formed by the absolute values $\sum |A_n|$ converges.

By virtue of the uniqueness theorem we can, therefore, write in this case:

$$f(x) = \sum A_n e^{i\Lambda_n x} .$$

For the proof of this theorem we consider again the Fejér kernel $K_n(t)$ in the simplest case where $n = 2$ Thus,

$$K_2(t) = 1 + \tfrac{1}{2}(e^{-it} + e^{it}) \qquad (= 1 + \cos t).$$

Now let v_n be the "phase" of the nth coefficient A_n, or $A_n = |A_n| e^{iv_n}$. We form for any value of N a "composite" kernel which is, indeed, the product not of the kernels $K_2(\Lambda_n t)$, but of the kernels resulting from the phase translation $K_2(\Lambda_n t + v_n)$. Thus we form the kernel,

$$\mathsf{K}_N(t) = K_2(\Lambda_1 t + v_1) \cdot K_2(\Lambda_2 t + v_2) \ldots K_2(\Lambda_N t + v_N).$$

By multiplying it out we obtain,

$$\mathsf{K}_N(t) = 1 + \tfrac{1}{2}\{e^{-iv_1} e^{-i\Lambda_1 t} + e^{-iv_2} e^{-i\Lambda_2 t} + \cdots + e^{-iv_N} e^{-i\Lambda_N t}\} + R(t),$$

where $R(t)$ is a finite trigonometric sum $\sum a_n e^{i\lambda_n t}$ whose exponents λ_n are (because of the linear independence of Λ_n) distinct from the numbers $0, -\Lambda_1, -\Lambda_2, \ldots, -\Lambda_n, \ldots$

As in the proof of the approximation theorem, we form the folded function $M\{f(x+t)\mathsf{K}_N(t)\}$, but only for $x = 0$, i.e.,

we form the mean value

$$\underset{t}{M}\{f(t)\,K_N(t)\}.$$

Since

$$\underset{t}{M}\{f(t)\,e^{-i\lambda t}\} = \begin{cases} A_n \text{ for } \lambda = \Lambda_n, \\ 0 \text{ for } \lambda \neq \Lambda_n \end{cases}$$

it follows that

$$\underset{t}{M}\{f(t)\,K_N(t)\} = \tfrac{1}{2}\{A_1 e^{-iv_1} + A_2 e^{-iv_2} + \cdots + A_N e^{-iv_N}\}$$
$$= \tfrac{1}{2}\{|A_1| + |A_2| + \cdots + |A_N|\}.$$

But $K_N(t) \geqq 0$ for all t, and $M\{K_N(t)\} = 1$. Thus the left side is at most equal to Γ if we denote the upper bound of $|f(t)|$ by Γ. We thus obtain the inequality

$$|A_1| + |A_2| + \cdots + |A_N| \leqq 2\Gamma,$$

whereby the absolute convergence of the series $\sum A_n$ is demonstrated.

It should be noted, as a matter of orientation that in the relation $\sum |A_n| \leqq 2\Gamma$ the factor 2 has no real significance; it is there only because we used a kernel of the second order in our proof. If we had used in place of $K_2(t)$ the kernel

$$K_p(t) = 1 + \frac{p-1}{p}(e^{-it} + e^{it}) + \cdots$$

we would have been led to the relation $\sum |A_n| \leqq \frac{p}{p-1}\Gamma$, and from that, (by taking the limit as $p \to \infty$), we should have found that $\sum |A_n| \leqq \Gamma$. <u>In this relation the equality sign alone is true</u> since obviously the upper bound of $|\sum A_n e^{i\Lambda_n x}|$ can not be greater than the upper bound of $\sum |A_n|$.

Because of the convergence of $\sum |A_n|$ we have just shown that

$$\underset{-\infty < x < \infty}{\text{upper limit}}\ |f(x)| = \sum |A_n|$$

which could have been proved from a well known theorem of Kronecker in Diophantine Approximations. This Kronecker theorem was used in the original proof of the convergence theorem.

APPENDIX I

GENERALIZATIONS OF ALMOST PERIODIC FUNCTIONS

94. The first papers on almost periodic functions were the basis for a number of further papers. Some of these are on various generalizations of this class of functions. The first such generalizations are due to Stepanoff; independently of him, Wiener in his important investigations on Fourier integrals was also led to one of Stepanoff's classes of functions. A very far reaching generalization was later given by Besicovitch. Finally, Weyl gave a generalization of the concept of almost periodicity different from the above mentioned ones. He has investigated his generalized class of functions by his own methods, at the end of his interesting work "Integralgleichungen und fast-periodische Funktioners" (Integral Equations and Almost Periodic Functions), in which new proofs are given of the main theorems of the ordinary theory of almost periodic functions. Some interesting contributions to the generalized theory by R. Schmidt, Kowanko, Franklin and Ursell, should also be mentioned.

These various generalizations and others related to them have recently been the subject of a detailed investigation by Besicovitch and the author. This brief appendix will be concerned chiefly with the general viewpoints which we took in that investigation in order to arrive at a unified treatment of the entire complex of questions.

95. First of all, as an introduction to the subject, let us say a few words on the theory "proper" of almost periodic functions which was expounded in detail in the preceding lectures.

By the closure of a set Φ of functions $f(x)$ $(-\infty < x < \infty)$ we mean the set $H(\Phi)$ of functions which is obtained from the given set Φ by extending it, adding to it all those functions which can be approximated uniformly for all x by functions of the set Φ.

In the following, E will throughout denote the set of

all finite sums of pure vibrations

$$s(x) = \sum_{n=1}^{N} a_n e^{i\lambda_n x}.$$

Furthermore, let F denote the set of all almost periodic functions; all continuous functions $f(x)$ with the property that for each $\varepsilon > 0$ there exists a relatively dense set of numbers (translation numbers) $\tau = \tau(\varepsilon)$ such that

$$|f(x+\tau) - f(x)| \leqq \varepsilon, \qquad -\infty < x < \infty.$$

The main theorem of the theory of almost periodic functions (the approximation theorem) then states that the set F is precisely the closure of E, thus

$$F = H(E). \tag{1}$$

By means of this equation (as stressed in the lectures) our class of functions is characterized by two different types of properties: on the one hand by vibration properties as the closure of finite sums of pure vibrations, on the other hand by translation properties (structural properties) as almost periodic functions.

To every almost periodic function belongs a Fourier series $\sum A_n e^{i\Lambda_n x}$ which in turn uniquely determines the function $f(x)$. Bochner's summation theorem, which extends to almost periodic functions Fejer's summation theorem for the Fourier series ofpurelyperiodic functions, furnishes an algorithm which leads from the Fourier series to uniformly approximating sums $s(x)$.

96. The task of generalizing the theory of almost periodic functions will include above all the generalization of the approximation theorem, i.e., of the above equation (1). There one can proceed in two different ways. One may either start with the left side of the equation, i.e., try to directly generalize the definition of almost periodicity, chiefly by abandoning the requirement of continuity and dealing with functions which are only subject to such conditions as measurability or integrability in the Lebesgue sense This was the procedure of, e.g., Stepanoff, who was the first to attack the problem of generalization, and with great success. The task in this case is to study the vibration prop-

erties of the generalized almost periodic functions defined in this way. Or conversely, one may start with the right side of equation (1) (that is, with the vibration properties) by generalizing the concept of the closure of the class $\boldsymbol{E}$, i.e., by replacing everywhere uniform convergence with another concept of limit. One is here confronted with the converse task: to investigate the corresponding generalized almost periodic properties of the closure defined in that manner, of the set of all finite sums. This was Besicovitch's procedure. In our opinion the latter point of view is the more natural one for a systematic treatment of the generalized theory of almost periodic functions. The problem here is quite clear and unambiguous; one considers various limiting processes G, forms each time the corresponding closure $H_G(\boldsymbol{E})$ and then seeks to characterize this class of functions by translation properties.

97. In the following I shall for brevity's sake confine myself to functions of class L, i.e., the class of all functions $f(x)$ $(-\infty < x < \infty)$, which are Lebesgue integrable in every finite interval. (Besicovitch and I, by the way, have investigated all classes L^p where p is any number $\geqq 1$).

In the theory of purely periodic functions of class L, considered on a finite interval (a, b) the following well known concept of limit is adopted: $\lim f_n(x) = f(x)$ shall mean that the mean value

$$\frac{1}{b-a}\int_a^b |f(x) - f_n(x)|\,dx$$

tends to 0 for $n \to \infty$. Or, expressed differently, we introduce a notion of distance of two functions in the well known fashion by saying that the distance between $f(x)$ and $g(x)$ is defined by

$$D[f(x), g(x)] = \frac{1}{b-a}\int_a^b |f(x) - g(x)|\,dx$$

and then the concept of limit $\lim f_n(x) = f(x)$ is defined simply by

$$D[f(x), f_n(x)] \to 0 \quad \text{for} \quad n \to \infty .$$

In the theory of almost periodic functions we always

deal with the infinite interval $-\infty < x < \infty$. If we want to extend the above concept of limit, or rather the above concept of distance, from a finite to an infinite interval we are faced with the choice of several different possibilities each of which has its special characteristics and its special interest. We introduce three such concepts of distance and denote them by $D_S[f(x), g(x)]$, $D_B[f(x), g(x)]$ and $D_W[f(x), g(x)]$ because they are very closely connected with the generalizations of almost periodic functions by Stepanoff, Besicovitch and Weyl.

Stepanoff's concept of distance is given by

$$D_S[f(x), g(x)] = \underset{-\infty<x<\infty}{\text{upper bound}} \frac{1}{L}\int_x^{x+L} |f(\xi) - g(\xi)|\, d\xi.$$

There L is a fixed positive number; its value is irrelevant (it may be taken, e.g., $= 1$) because this concept of distance defines a concept of limit which does not depend on the particular value of L, as can easily be seen.

With Besicovitch's concept of distance $D_B[f(x), g(x)]$ the mean value is extended immediately over the whole interval $-\infty < x < \infty$:

$$D_B[f(x), g(x)] = \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x) - g(x)|\, dx.$$

Finally, Weyl's concept of distance is intermediate between the two; we here consider, to start with, a fixed length L, as in Stepanoff's case, but let it afterward grow without bound:

$$D_W[f(x), g(x)] = \lim_{L\to\infty} \underset{-\infty<x<\infty}{\text{upper bound}} \frac{1}{L}\int_x^{x+L} |f(\xi) - g(\xi)|\, d\xi.$$

It is easy to prove that the limit as $L\to\infty$ really exists.

For each of the three concepts of distance, the corresponding concept of limit

$$S\text{-}\lim f_n(x) = f(x), \quad B\text{-}\lim f_n(x) = f(x), \quad W\text{-}\lim f_n(x) = f(x)$$

is defined by the relation

$$D_S[f(x), f_n(x)] \to 0, \quad D_B[f(x), f_n(x)] \to 0, \quad D_W[f(x), f_n(x)] \to 0,$$

and by using, each time, the corresponding concept of

limit, the closure of the set $\boldsymbol{E}$ is formed, that is,

$$H_S(\boldsymbol{E}), \quad H_B(\boldsymbol{E}), \quad H_W(\boldsymbol{E}).$$

as the case may be.
As is easy to see, we then have

$$H_S(\boldsymbol{E}) \subset H_W(\boldsymbol{E}) \subset H_B(\boldsymbol{E}).$$

98. Our problem now is to characterize each of the three classes of functions $H_S(\boldsymbol{E}), H_B(\boldsymbol{E}), H_W(\boldsymbol{E})$ thus defined, by means of translation properties. This is the very same problem as the one solved by equation

$$\text{(1)} \qquad \boldsymbol{H}(\boldsymbol{E}) = \boldsymbol{F}$$

for the closure $\boldsymbol{H}(\boldsymbol{E})$, this closure being defined by uniform convergence.

By an extremely simple but fundamentally important consideration, this problem can be reduced to the corresponding problem of the original theory, which is solved by equation (1). By noting the fact that uniform convergence is the narrowest of all limiting processes considered -- i.e., that the closure $\boldsymbol{H}(\boldsymbol{E})$ obtained with uniform convergence is the smallest of all closures $H(\boldsymbol{E})$ we can use the approximation theorem (1) of the original theory directly as a valid result in building up the generalized theories, without having to come back to the proof of this theorem. For, from the inequality

$$\boldsymbol{E} \subset \boldsymbol{H}(\boldsymbol{E}) \subset H_G(\boldsymbol{E}),$$

where G denotes any one of the above three limiting processes S-, B- or W-lim, it follows immediately that $H_G(\boldsymbol{E}) \subset H_G[\boldsymbol{H}(\boldsymbol{E})] \subset H_G[H_G(\boldsymbol{E})]$ and hence, using $H_G[H_G(\ldots)] = H_G(\ldots)$, that

$$H_G(\boldsymbol{E}) = H_G[\boldsymbol{H}(\boldsymbol{E})];$$

but because of $\boldsymbol{H}(\boldsymbol{E}) = \boldsymbol{F}$, this means that

$$\text{(2)} \qquad H_G(\boldsymbol{E}) = H_G(\boldsymbol{F}).$$

This last equation however shows that the problem of characterization of the class $H_G(\boldsymbol{E})$ of functions is simply equivalent to investigating the effect of the respective limiting processes G as applied to the almost periodic functions proper of the original class $\boldsymbol{F}$. We thus do not even have to deal any more with the class $\boldsymbol{E}$, i.e., with the theory of vibrations; the connection between vibrations and translation

properties is given once and for all by the original equation (1) ; whatever remains to be investigated is concerned only with pure translation properties.

99. As to the results obtained, i.e., as to the actual characterization of the above three classes of functions $H_S(\boldsymbol{E})$, $H_B(\boldsymbol{E})$, $H_W(\boldsymbol{E})$ by means of translation properties, we shall confine ourselves in this short survey to the consideration of Stepanoff's class $H_S(\boldsymbol{E})$. For the other classes of functions the results are of a much more complicated nature (and the proofs are much more difficult, especially for Besicovitch's class).

By using the concept of distance $D_S[f(x), g(x)]$ introduced above, the main result of Stepanoff's theory may be stated in an exceedingly simple way, closely analogous to the main result of the theory of almost periodic functions proper as expressed by equation (1). For it is not difficult to demonstrate that this equation (1) remains valid if on both sides of the equation the primitive (and implicitly adopted) concept of distance

$$\boldsymbol{D}[f(x), g(x)] = \underset{-\infty<x<\infty}{\text{upper bound}} |f(x) - g(x)|,$$

which corresponds to the concept of limit of <u>everywhere uniform convergence</u>, is replaced by the Stepanoff concept of distance $D_S[f(x), g(x)]$. Speaking precisely, we can put it this way: If we use the expression Stepanoff <u>translation number</u> associated with ε of a function $f(x)$ (of class L) for every number $\tau = \tau_f(\varepsilon)$ for which

$$D_S[f(x+\tau), f(x)] \leqq \varepsilon$$

and if we call almost periodic in Stepanoff's sense, every function $f(x)$, having for each $\varepsilon > 0$ a relatively dense set of Stepanoff translation numbers $\tau_f(\varepsilon)$, then the set F_S of all these functions is simply the above closure $H_S(\boldsymbol{E})$, i.e., we have, in complete analogy to equation (1) , the fundamental equation

$$F_S = H_S(\boldsymbol{E}).$$

100. So far in our discussion of the generalized almost periodic functions we have not even mentioned the concept of the "Fourier series" of the functions in question. In the

theory of the almost periodic functions proper, that concept formed the basis of the entire proof of the main theorem, i.e., of the above equation (1). That a complete avoidance of this concept in the preceding considerations was possible, is linked very closely with the leading idea of the whole investigation, this idea being to extend theorems directly from the original theory to the generalized classes of functions in finished form (without going into their proofs).

However, in the further developments of the theory of the generalized almost periodic functions, the concept of the Fourier series regains its place, now in the most natural connection: when it is a question of finding an algorithm which can furnish, for a given function $f(x)$ almost periodic in the generalized sense, finite trigonometric sums $s(x)$ which converge toward the function $f(x)$ in the sense of convergence appropriate to the adopted concept of distance.

Fourier series are the best imaginable means for obtaining such an algorithm. First of all it results that with every function of any one of the generalized classes of functions considered, there can actually be associated a Fourier series $\sum A_n e^{i\Lambda_n x}$. This follows simply from the fact that the mean value theorem of the theory of the almost periodic functions proper can be extended to the generalized almost periodic functions. It results further -- still for an arbitrary function $f(x)$ of any one of the generalized classes of functions -- that the Fejer-Bochner method of summation achieves exactly the same result here as it does in the theory of the almost periodic functions proper: it furnishes, starting from the Fourier series of the function $f(x)$ finite sums $s(x)$ which converge toward the given function $f(x)$ in the sense appropriate to the concept of distance adopted.

101. On the basis of the fact just mentioned -- namely, that it is possible to give an algorithm which leads from the Fourier series $\sum A_n e^{i\Lambda_n x}$ of any one of the generalized almost periodic functions $f(x)$ to approximating sums $s(x)$ -- it might seem at first as if one should be able to infer immediately

that the function $f(x)$ is uniquely determined by its Fourier series in each of the generalized cases, just as in the case of the almost periodic functions proper. A uniqueness theorem in this strong form does not, however, hold for the generalized almost periodic functions (this is already true for the Lebesgue integrable purely periodic functions). Indeed, for the limiting processes S-lim, W-lim, B-lim (as opposed to the ordinary limiting process), the limit function $f(x)$ of a sequence of approximating functions $f_n(x)$ is not uniquely determined in the strict sense, but uniquely determined only up to a "null function", i.e., a function $h(x)$ for which the respective distances $D_S[f(x), 0]$, $D_W[f(x), 0]$ and $D_B[f(x), 0]$ are equal to zero Only if such functions $f(x), g(x), \ldots,$ which differ only by null functions, are in each case considered as "identical", will the "strong" uniqueness theorem hold again.

In the case of Stepanoff's concept of distance, as is easy to see, a function $h(x)$ is a null function only if it vanishes everywhere but on a set of Lebesgue measure zero. A function almost periodic in Stepanoff's sense is thus determined by its Fourier series "almost everywhere". In the two other cases, the domain of null functions comprises much more, so that there holds no such strong uniqueness theorem.

102. In closing we add an important remark concerning Besicovitch's generalization of the almost periodic functions. If instead of functions of class L we consider functions of class L^2 (i.e., the class of all functions $f(x)$ which, together with their squares, are Lebesgue integrable in every finite interval), then for the Besicovitch functions the theorem analogous to the classical Riesz-Fischer theorem for purely periodic functions of class L^2 will hold. Besicovitch's theorem is that the necessary and sufficient condition for an arbitrary trigonometric series $\sum a_n e^{i\lambda_n x}$ to be the Fourier series of a function almost periodic in Besivocitch's sense is simply that the series $\sum |a_n|^2$ should converge. With this theorem, the generalized theory of almost periodic functions is, in a certain sense, completed.

The reader who wishes to gain a deeper knowledge of the problems discussed in this appendix is referred, e.g., to

the recent book "Almost Periodic Functions" by Besicovitch which deals in detail with the generalizations of almost periodic functions.

APPENDIX II

ALMOST PERIODIC FUNCTIONS OF A COMPLEX VARIABLE

103. This appendix is a brief report on the theory of analytic almost periodic functions (developed in my third paper in the Acta Math.).

We start here with a complex variable $z = re^{i\theta}$ and consider the non-denumerable infinity of powers $z^{\lambda} = r^{\lambda} e^{i\lambda\theta}$ where λ goes through all real values from $-\infty$ to $+\infty$. We consider these powers on the Riemann surface L: $0 < r < \infty$ $-\infty < \theta < \infty$ of infinitely many sheets, i.e., on the logarithmic surface with the two branch points of order infinity at $z = 0$ and $z = \infty$. Our problem then is:

How must an analytic function $f(z)$, regular on a circular ring $r_1 < r < r_2$ $-\infty < \theta < \infty$ with infinitely many sheets on the surface L, be constituted in order for $f(x)$ to be representable as a sum of denumerably many powers z^{λ_n}, i.e., for $f(x)$ to admit of a meaningful representation of the form

$$\sum a_n z^{\lambda_n}$$

on the entire circular ring (i.e., on all sheets at the same time, not only on every partial region made up of finitely many sheets of the ring)?

Of special interest is the case $r_1 = 0$ (or the corresponding case $r_2 = \infty$) where our problem specializes into the question: how must a function behave at a logarithmic branch point if it admits of an "irregular" Laurent series $\sum a_n z^{\lambda_n}$ at that point?

In order to connect this problem as directly as possible with the theory of almost periodic functions as given in the preceding lectures -- which theory will form the real basis for the theory to be developed in the present case -- it will be convenient in the following to go from the Riemann surface L to the "schlicht" plane of the variable $s = \sigma + it$ by means of the simple transformation $z = e^s$ (or $s = \log z$).

The problem then reads:

Which analytic functions $f(s)$ regular in a strip $\sigma_1 < \sigma < \sigma_2$ can be developed into a series of the form

$$\sum a_n e^{\lambda_n s},$$

i.e., into a so-called "Dirichlet series"?

To the above case $r_1 = 0$ (or $r_2 = \infty$) corresponds here the case $\sigma_1 = -\infty$ (or $\sigma_2 = +\infty$) in which case the strip $\sigma_1 < \sigma < \sigma_2$ is extended to a whole left (or right, respectively) half-plane.

We wish at once to stress that our concept of a "Dirichlet series" differs from the usual one in two essential points. Firstly, we do not start with series written down in an arbitrary way, but we admit only such series as are generated by a function; secondly, the real exponents λ_n are not subject to any condition whatever in our case, whereas usually only such sequences of exponents are considered as do not have a finite point of accumulation. This second remark is of decisive importance for the whole theory, just as in the case of a real variable; only the dropping of all restrictions on the sequences of exponents makes possible a simple and easily understandable characterization of the class of functions which can be developed into Dirichlet series.

104. As we shall see, the theory of almost periodic functions of a complex variable shares some essential features with the theory of almost periodic functions of a real variable. One part of the new theory however, follows an entirely different direction, being based on the added premise of the analytic character of the functions.

The basic definition here is the following:

An analytic function regular in a strip $\alpha < \sigma < \beta$ $(-\infty \leqq \alpha < \beta \leqq +\infty)$ is called "almost periodic in (α, β)" if for every ε there exists a length $l = l(\varepsilon)$ such that every interval $a < t < a + l$ of length l on the imaginary axis contains at least one translation number $\tau = \tau(\varepsilon)$ associated with ε, i.e., a number τ satisfying the inequality

$$|f(s + i\tau) - f(s)| \leqq \varepsilon$$

<u>for all</u> s <u>in the entire strip</u> $\alpha < \sigma < \beta$.

The requirements made on the function are thus seen to be: firstly, on every vertical straight line $\sigma = \sigma_0$ of the strip $\alpha < \sigma < \beta$ it should be an almost periodic function of the (real) ordinate t, and secondly, the almost periodicity should take place "uniformly" on the various straight lines.

As can be shown by a somewhat complicated method of construction, there actually exist analytic functions $f(s)$ almost periodic on every straight line of a vertical strip but lacking the above-mentioned "uniform" almost periodicity on the different straight lines. A detailed investigation of these rather peculiar functions will be published soon by the Danish mathematician R. Petersen.

Every purely periodic function $f(s)$ in a strip will, of course, also be almost periodic in that strip. Hence in particular every "<u>pure vibration</u>" $ae^{\lambda s}$ where λ denotes an arbitrary real number, is almost periodic in the whole plane $(-\infty, +\infty)$.

It is convenient for the exposition to introduce, besides the above concept "almost periodic in (α, β)", also the concept "almost periodic in $[\alpha, \beta]$"; this to denote simply that $f(s)$ is almost periodic in the partial strip $(\alpha + \delta, \beta - \delta)$ for every $\delta > 0$.

105. The theorem that every function $f(s)$ almost periodic in (α, β) is <u>bounded</u> in $[\alpha, \beta]$ (i.e., in every partial strip) holds in the present case too, just as it holds for almost periodic functions of a real variable. The following theorem, a kind of <u>converse</u> of the preceding one, is of great importance: If we know of a function $f(s)$ regular in the strip (α, β) only that it is almost periodic on <u>one single line</u> $\sigma = \sigma_0$ of the strip (i.e., that the function $f(\sigma_0 + it) = F(t)$ is an almost periodic function of t), then $f(s)$ will be almost periodic in the whole strip $[\alpha, \beta]$, provided it is bounded in $[\alpha, \beta]$. This theorem permits the introduction of the concept of a "maximal strip" in which a function $f(s)$ is almost periodic, i.e., the largest strip $[\alpha^*, \beta^*]$, that contains the given strip $[\alpha, \beta]$ in which $f(s)$ is almost periodic and which is such that in it the function is bounded (and therefore almost periodic).

The above-mentioned theorem also furnishes us with a very convenient means for transferring complete theorems from the theory of almost periodic functions of a real variable directly to that of the complex variable, without going into the proofs a second time. Thus by means of that theorem the following can, for instance, be proved at once: The sum of two functions almost periodic in a strip $[\alpha, \beta]$, is itself almost periodic in $[\alpha, \beta]$. For $f(s)$ and $g(s)$, and hence also $f(s)+g(s)$ are bounded in $[\alpha, \beta]$; and on an arbitrarily chosen fixed straight line $\sigma=\sigma_0$, $f(\sigma_0+it)$ and $g(\sigma_0+it)$ and hence also $f(\sigma_0+it)+g(\sigma_0+it)$ are almost periodic functions of t. Also the product, and the limit function of a uniformly convergent sequence of almost periodic functions are themselves almost periodic; hence, in particular, every series of the form

$$\sum a_n e^{\lambda_n s}$$

which is uniformly convergent in $[\alpha, \beta]$, represents a function almost periodic in $[\alpha, \beta]$ Furthermore -- as in the case of real variables - the theorem holds true that the integral of an almost periodic function is itself almost periodic provided merely that it stays bounded. But the two following theorems are new and without analogue in the real variable case: The derivative of an almost periodic function is itself almost periodic, and (essential for various investigations) the quotient $f(s)/g(s)$ of two almost periodic functions is always almost periodic provided only that the denominator $g(s)$ has no zeros in the strip; it is therefore, enough to assume only that $g(s) \neq 0$, instead of assuming $|g(s)|>k>0$; in fact it is a characteristic property of an analytic almost periodic function $g(s)$ that if it is $\neq 0$ throughout (α, β), then the absolute value $|g(s)|$ must have a positive lower bound in every partial strip.

It can be shown in fact that for the almost periodicity of the quotient $h(s)=f(s)/g(s)$ of two almost periodic functions $f(s)$ and $g(s)$ it is sufficient that the function $h(s)$ be regular in the strip in question; in other words, it is quite permissible for the denominator $g(s)$ to vanish at certain points, provided only that the numerator $f(s)$ has zeros (of at least the same multiplicity) at the same points.

This theorem, which apparently forms a starting point for the development of a theory of the meromorphic almost periodic functions, can be thought of as a generalization of a beautiful theorem by Ritt on exponential polynomials.

106. In order to get to the Dirichlet expansion $\sum A_n e^{\Lambda_n s}$ of a function $f(s)$ almost periodic in (α, β), we proceed as follows. On every fixed straight line $\Re(s) = \sigma$ of the strip, $f(\sigma + it) = F_\sigma(t)$ is almost periodic in t and hence has a Fourier series

$$F_\sigma(t) \sim \sum A_n^{(\sigma)} e^{i\Lambda_n^{(\sigma)} t} .$$

Now it can be shown easily, using Cauchy's integral theorem, that in the first place the exponents $\Lambda_n^{(\sigma)}$ are independent of σ, and secondly that the coefficients $A_n^{(\sigma)}$ depend on σ in the "proper" way, i.e., $A_n^{(\sigma)} = A_n e^{\Lambda_n \sigma}$ where A_n is a constant not depending on σ. The Fourier series on the non-denumerably infinity of straight lines $\Re(s) = \sigma$ thus combine into a single expansion of the form

$$\sum A_n e^{\Lambda_n \sigma} \cdot e^{i\Lambda_n t} = \sum A_n e^{\Lambda_n s}$$

and it is this expansion which we call the "Dirichlet series" of the function $f(s)$ in the strip (α, β). We write

$$f(s) \sim \sum A_n e^{\Lambda_n s} .$$

In the special case of a purely periodic function of period $2\pi i$ (or, if we return to the z-plane, of a function in the schlicht circular ring $r_1 < |z| < r_2$) our Dirichlet series becomes the ordinary Laurent series of the function $f(s)$.

107. Two theorems that carry over immediately into the present case from the case of a real variable are the uniqueness theorem which states that two different functions almost periodic on the same strip have associated with them two different Dirichlet series, and the theorem on convergence in the mean, i.e., the Parseval equation

$$M\{|f(\sigma + it)|^2\} = \sum |A_n|^2 e^{2\Lambda_n \sigma}, \qquad \alpha < \sigma < \beta .$$

Also the rules of calculation for operations with the series expansions are extended immediately: The Dirichlet

expansion of a sum, of a product, of a limit function, of an integral (if it stays bounded) and of a derivative is obtained by formal calculation from the Dirichlet series of the given functions.

The approximation theorem also carries over without difficulty and yields the following main theorem of the whole theory: For a function analytic in $[\alpha, \beta]$ to be almost periodic, it is necessary and sufficient that it can be approximated by means of exponential polynomials

$\sum_1^N a_n e^{\lambda_n s}$ uniformly in $[\alpha, \beta]$. Approximating exponential polynomials can be found again by Bochner's summation method.

Examples of Dirichlet series which converge uniformly in the strip $[\alpha, \beta]$, are furnished by the case of linearly independent exponents. Further simple examples are obtained by means of the following theorem which acts as a bridge to the theory of ordinary Dirichlet series whose sequences of exponents have no finite point of accumulation: If the series

$$\sum e^{-|\Lambda_n|\delta}$$

converges for every $\delta > 0$, then the Dirichlet expansion $\sum A_n e^{\Lambda_n s}$ is absolutely convergent in the whole strip (α, β) of almost periodicity. The ordinary Laurent expansion $\sum a_n e^{n\frac{2\pi}{p}}$, of purely periodic functions comes under this as a special case.

108. For ordinary Laurent series $\sum_{-\infty}^{\infty} a_n z^n$ the special case where all exponents are non-negative, i.e., the case of an ordinary power series $\sum_0^{\infty} a_n z^n$, plays a special role. In the same way for the present theory, a special interest attaches to the case where the Dirichlet exponents Λ_n of our almost periodic function $f(s)$ are all of the same sign. We therefore proceed to the consideration of such functions.

One arrives here at results which are almost as

rounded out as those obtained for the classical special case of the proper power series $\sum_0^\infty a_n z^n = \sum_0^\infty a_n e^{ns}$ (only that convergence in the usual sense in our general case need not of course occur). We begin with an important boundary value theorem the proof of which is based on the above approximation theorem and uses the classical Phragmen-Lindelof principle.

Let

$$F(t) \sim \sum A_n e^{i\Lambda_n t}$$

be an entirely arbitrary almost periodic function (not even assumed to be analytic) with all its exponents Λ_n positive. Then it is the boundary function of an analytic function regular in the half-plane $\sigma < 0$, i.e., there exists a function $f(s)$, regular for $\sigma < 0$ and continuous for $\sigma \leqq 0$ which on the imaginary axis $\sigma = 0$ satisfies the boundary condition

$$f(it) = F(t)$$

and this function $f(s)$ is almost periodic in $(-\infty, 0)$, tends to 0, uniformly in t for $\sigma \to -\infty$, and its Dirichlet expansion is obtained from the Fourier expansion for $F(t)$, by replacing it by $\sigma + it = s$, i.e.,

$$f(s) \sim \sum A_n e^{\Lambda_n s}.$$

From this boundary value theorem follows a result on almost periodic functions of a real variable, which is surprising at first sight: such a function is completely determined by its values on an arbitrarily small interval if its Fourier exponents are all positive (or all negative).

From the above boundary value theorem we can derive at once the following important theorem.

If a function $f(s) \sim \sum A_n e^{\Lambda_n s}$ almost periodic in $[\alpha, \beta]$ has its Dirichlet exponents all positive, then it can be continued analytically over the whole left half-plane $\sigma \leqq \alpha$ and then represents a function $f(s)$. almost periodic in $(-\infty, \beta]$ which tends to 0 uniformly in t for $\sigma \to -\infty$.

Indeed, we have only to consider the almost periodic function $F(t) = f(\sigma_0 + it)$ for a fixed σ_0 in the interval

$\alpha < \sigma < \beta$. Its Fourier series is given by $\sum A_n e^{\Lambda_n \sigma_0} e^{i\Lambda_n t}$ and has, therefore, none but positive exponents; hence it is the boundary function of an almost periodic function analytic in the whole left half-plane $\sigma < \sigma_0$. But then it must coincide with the given function $f(s)$, because it coincides with that function on the line $\sigma = \sigma_0$.

<u>Corollary</u>: Every function $f(s) \sim \sum A_n e^{\Lambda_n s}$ which is almost periodic in a strip (α, β) and which has its exponents Λ_n bounded, $|\Lambda_n| < K$, must be an entire transcendental function. For $f(s)e^{Ks}$ has none but positive exponents and can, therefore, be continued toward the left, while $f(s)e^{-Ks}$ has none but negative exponents and can, therefore, be continued toward the right. Such functions with bounded exponents (but which may be everywhere dense in a finite interval) play a role in our theory similar to that played by polynomials in the theory of analytic functions in a <u>schlicht</u> circular ring.

If the assumption of "none but positive" exponents is weakened to only non-negative exponents, i.e., if a constant term c is admitted in the expansion $f(s) \sim \sum A_n e^{\Lambda_n s}$, then the above theorem will of course hold just the same, except that $f(s)$ will converge for $\sigma \to -\infty$, to just that constant term (instead of zero). On the other hand, one sees immediately that a function $f(s)$ almost periodic in a left half-plane $[-\infty, \beta]$ can not even remain bounded for $\sigma \to -\infty$ if its Dirichlet expansion contains at least one negative exponent Λ_n; for with fixed σ we have

$$\underset{-\infty<t<\infty}{\text{upper bound}} |f(\sigma + it)| \geq |M\{f(\sigma + it)e^{-i\Lambda_n t}\}| = |A_n e^{\Lambda_n \sigma}|,$$

where the right side tends to ∞ for $\sigma \to -\infty$, because of $\Lambda_n < 0$. Hence we can conclude: If a function almost periodic in $[-\infty, \beta]$ remains bounded for $\sigma \to -\infty$, then it must have only non-negative exponents and must, therefore, tend to a definite limit.

109. It follows from the above considerations that for any function $f(s)$ almost periodic in a half-plane $[-\infty, \beta]$

there are only three possibilities in approaching the "point" at infinity $\sigma = -\infty$, just as is the case, in accordance with Weierstrass theorem, for a purely periodic function (that is, for a function regular in the neighborhood of a point of the schlicht plane, if we return to the z-plane):

A. $f(s)$ converges to a finite limit (case of regularity).

B. $|f(s)|$ tends to ∞ (case of a pole).

C. The values of $f(s)$ lie everywhere dense in every half-plane $\sigma < \sigma_0$ (case of an essential singularity.)

Indeed, we can conclude the following, literally as in the proof of Weierstrass' theorem: If $f(s)$ does not come under case C, i.e., if there exists a number a with $|f(s) - a| > k > 0$, then we must have either case A or case B because the function $g(s) = 1/(f(s) - a)$ is almost periodic and bounded in a left half-plane, and therefore must tend to a limit, in accordance with the above. If this limit is $\neq 0$, we have case A; if it is $= 0$, we have case B.

In case C even Picard's theorem holds, i.e., the function $f(s)$ assumes all values with the exception of one at most, in every left half-plane.

But how can we recognize from the series expansion of the function whether we have case A, B or C? The answer is very simple (the proof however is not too easy):

a. The regular behavior A corresponds to the case where all Λ_n are $\geqq 0$.

b. The polar behavior B corresponds to the case in which negative exponents occur, amongst them a numerically largest one.

c. The essentially singular behavior C corresponds to the case in which there occur negative exponents without there being a numerically largest one among these.

Thus the last case comprises the functions whose exponents have $-\infty$ as a point of accumulation, as well as those functions for which the exponents have a finite negative number Λ^* as lowest point of accumulation that does not belong itself, however, to the set of the Λ_n. A more refined analysis however, shows up differences between these two classes of functions; thus it can be shown that for a

function $f(s)$ of the second class there exists no exceptional value in the sense of Picard's theorem at all, i.e., that $f(s)$ assumes all values in every left half-plane.

110. After this discussion of almost periodic functions that are defined over an entire half-plane, we return again to the consideration of almost periodic functions defined on an arbitrary vertical strip. Our aim is now to deal with the question of the Laurent separation of almost periodic functions. The classical Laurent theorem for a function analytic in a schlicht circular ring $r_1 < |z| < r_2$ or in our terminology, for a function $f(s)$, purely periodic in a strip $\sigma_1 < \sigma < \sigma_2$, states that $f(s)$ can be decomposed into two purely periodic summands

$$f(s) = f_1(s) + f_2(s)$$

such that $f_1(s)$ is analytic in the left half-plane $\sigma < \sigma_2$, and $f_2(s)$ is analytic in the right half-plane $\sigma > \sigma_1$, and such that $f_1(s)$ and $f_2(s)$ respectively are still regular at the "points" $\sigma = -\infty$ or $\sigma = +\infty$ respectively. After the above results on almost periodic functions it might seem at first as if the Laurent separation could be immediately transferred to an arbitrary function $f(s) \sim \sum A_n e^{\Lambda_n s}$ almost periodic in a strip $\sigma_1 < \sigma < \sigma_2$, by simply forming the two partial series

$$\sum_{\Lambda_n > 0} A_n e^{\Lambda_n s} \quad \text{and} \quad \sum_{\Lambda_n < 0} A_n e^{\Lambda_n s}$$

with only positive or only negative exponents respectively. These partial series would then, one might think, represent almost periodic functions $f_1(s)$ and $f_2(s)$ of which $f_1(s)$ could be continued analytically toward the left and $f_2(s)$ toward the right. But this method is not permissible offhand because it is not clear whether or not the two partial series

$\sum_{\Lambda_n > 0}$ and $\sum_{\Lambda_n < 0}$ are, each by itself, Dirichlet expansions of almost periodic functions. And indeed it is not only this idea for the proof but even the Laurent separation itself which may fail. It can be shown by means of suitably con-

structed examples that there exists, indeed, almost periodic functions for which a Laurent separation is not possible. This would at first seem rather discouraging, because, primarily a Laurent expansion would provide a powerful tool for further advances in the theory, besides being of interest in its own right. Fortunately however, a slight change of direction of thought can overcome this difficulty. For it can be shown that the derivative $f'(s)$ of an almost periodic function $f(s)$ always admits of a Laurent expansion, even if it itself does not. The analytic reason for this lies in the Cauchy representation

$$f'(s) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-s)^2}\,dz$$

where the second power in the denominator makes it possible to extend the path of integration to infinity; the corresponding result is in general not possible for the representation of $f(s)$ itself,

$$f(s) = \frac{1}{2\pi i}\int \frac{f(z)}{z-s}\,dz \quad .$$

It is also easily understandable from a purely formal viewpoint, a superficial consideration of the Dirichlet expansions of $f(s)$ and $f'(s)$, that $f'(s)$ behaves much more nicely at $\lambda = 0$ with respect to a separation than does $f(s)$ itself.

For in passing from the series expansion $\sum A_n e^{\Lambda_n s}$ for $f(s)$ to the expansion $\sum A_n \Lambda_n e^{\Lambda_n s}$ for $f'(s)$, the terms $A_n e^{\Lambda_n s}$ acquire factors Λ_n which are very small just in the neighborhood of $\lambda = 0$.

By means of the Laurent separation of the derivative one can in many cases actually obtain the same results as would be furnished by a separation (not always possible) of $f(s)$ itself. We cite e.g., the extended uniqueness theorem which states: If $f(s)$ and $g(s)$ are almost periodic in the strips (α, β) and (γ, δ) respectively, and if the Dirichlet expansion of $f(s)$ in (α, β) is formally identical with that of $g(s)$ in (γ, δ), then $f(s)$ and $g(s)$ are the same analytic function, even in the case where the two strips are entirely outside of each other, say $\alpha < \beta < \gamma < \delta$. There exists

then a function almost periodic in the entire strip $\alpha < \sigma < \delta$ which coincides with $f(s)$ in (α, β) and with $g(s)$ in (γ, δ). The proof is quite analogous to the proof of the corresponding theorem for ordinary Laurent series; only in our case there is first established the identity of the derivatives $f'(s)$ and $g'(s)$ by means of Laurent separation, and after that we go over to the functions $f(s)$ and $g(s)$ themselves through integration. (A more direct proof incidentally, was later discovered of this extended existence theorem).

111. We consider again the case where $f(s)$ is given in an entire half-plane. We shall discuss the so-called inversion theorem of the theory which is of particular interest because it expresses in a peculiar way the apparent "completeness" of the class of almost periodic functions. In order to bring out distinctly the analogy with the corresponding theorem for the special case of an ordinary power series (and with the case, seemingly, but not really, more general, of an algebraic singularity), we prefer to operate in the z-plane instead of the s-plane (i.e., to perform the transformation of variables $s = \log z$ mentioned at the beginning). We then have to deal with a function $\varphi(z) = f(\log z)$ -- instead of a function $f(s)$ defined in a left half-plane -- which is analytic in a certain circular neighborhood of the branch point $z = 0$ of infinite order and is also almost periodic there with respect to the angles θ. Its Dirichlet expansion -- or perhaps it is better here to speak of a "generalized Laurent expansion" -- is denoted by

$$\sum A_n z^{\Lambda_n}$$

We assume that the exponents Λ_n are all positive (regular case), and besides, that there is a smallest among them. It can then be easily seen that our function $w = \varphi(z)$ maps a sufficiently small neighborhood of the branch point $z = 0$ of infinite order onto a certain full neighborhood of the branch point $w = 0$ of infinite order. Thus we can speak of the inverse function $z = \psi(w)$ of $w = \varphi(z)$ in a neighborhood of this latter branch point $w = 0$. It can be shown furthermore -- and this is the inversion theorem -- that this inverse function $z = \psi(w)$ is itself an almost periodic function.

From this theorem it follows e.g., that the inverse

function of Riemann's zeta function $\zeta(s)=\sum_{1}^{\infty}\frac{1}{n^s}$ as well as of other important number-theoretic functions are themselves almost periodic functions (after a harmless transformation of variables is applied to them). It might be an interesting task to investigate in more detail these almost periodic inverse functions and in particular, their Dirichlet expansions.

112. I conclude this short sketch of the theory of almost periodic functions of a complex variable with a discussion of some interesting and still unpublished results of Jessen, concerning the distribution of values of an almost periodic function. We will use again the language of the s-plane and consider any function $f(s)$ almost periodic in a strip $[\alpha, \beta]$. Let a be a fixed value which is assumed by $f(s)$ in this strip. We want to know the distribution of all a-places in our strip. Without less of generality we may take $a=0$ so that we deal just with the zeros of $f(s)$. Let (γ, δ) be a partial strip of (α, β) which contains at least one of the zeros of $f(s)$ in its interior. We denote by $N(T)$ the number of zeros in that part of the strip (γ, δ) whose ordinates lie between $-T$ and T. It follows without difficulty from the almost periodicity of $f(s)$ that this number $N(T)$ grows to infinity for $T \to \infty$, and just as in the purely periodic case it does so with order T in the sense that the quotient $N(T)/2T$ remains between two positive constants for sufficiently large values of T. There arises quite naturally the further question as to whether the distribution of zeros in (γ, δ) is also (as in the purely periodic case) such a regular one that the above quotient $N(T)/2T$ even tends to a definite limit $G=G(\gamma, \delta)$ for $T \to \infty$. In that case, one could again speak of a certain "relative frequency" of the zeros in (γ, δ). The answer is that for a given almost periodic function $f(s)$ this is not always but "almost always" the case, in the sense that there are at most denumerably many pairs of values γ, δ in the interval $\alpha < \sigma < \beta$ for which the limit $G(\gamma, \delta)$ does not exist. The critical exceptional values of γ and δ that may exist, as well as the limit $G(\gamma, \delta)$ for all other pairs of values γ, δ can be characterized in the following simple fashion, ac-

cording to Jessen: For every fixed σ of the interval $\alpha < \sigma < \beta$ one forms the mean value

$$\varphi(\sigma) = \underset{t}{M}\{\log|f(\sigma + it|\},$$

whose existence may be deduced from the almost periodicity of $f(s)$. The real function $\varphi(\sigma)$ (on $\alpha < \sigma < \beta$) associated with our analytic function $f(s)$ turns out to be continuous and convex so that the curve given by $\varphi = \varphi(\sigma)$ possesses a definite tangent except for at most denumerably many points where it has corners. This convex curve $\varphi = \varphi(\sigma)$ governs the distribution of zeros of $f(s)$ in the following simple manner: the abcissas σ corresponding to the corners of the curve are exactly the above-mentioned possible exceptional values of γ and δ., whereas for every other pair of values γ, δ, the derivatives $\varphi'(\gamma)$ and $\varphi'(\delta)$ both exist; so, then, does the above limit $G(\gamma, \delta)$ which is easily determined from the formula

$$G(\gamma, \delta) = \frac{1}{2\pi}\{\varphi'(\delta) - \varphi'(\gamma)\}.$$

On closer examination this theorem turns out to be a natural generalization to almost periodic functions of a classical formula by Jensen. It settles in a general way the problem of the distribution of values of almost periodic functions. But when it is a question of investigating the distribution of values for a definite given almost periodic function, and in particular, of determining more definitely the above limit $G(\gamma, \delta)$, one must usually resort to the Dirichlet expansion of the function. For the case of Riemann's zeta function and of some related functions there exist for a number of years detailed investigations of this kind (which, incidentally, have recently been much refined by Jessen and the author). We shall, however not here enter more closely upon these rather special investigations, which were the starting point of the whole theory of almost periodic functions.

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